NONCOMMUTATIVE RIESZ TRANSFORMS – A PROBABILISTIC APPROACH

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Abstract. For $2 \le p < \infty$ we show the lower estimates

$$\|A^{\frac{1}{2}}x\|_p \;\le\; c(p)\max\{\;\|\Gamma(x,x)^{\frac{1}{2}}\|_p,\;\|\Gamma(x^*,x^*)^{\frac{1}{2}}\|_p\}$$

for the Riesz transform associated to a semigroup (T_t) of completely positive maps on a von Neumann algebra with negative generator $T_t = e^{-tA}$, and gradient form

$$2\Gamma(x,y) \;=\; Ax^*y + x^*Ay - A(x^*y)\;.$$

As additional hypothesis we assume that $\Gamma^2 \;\ge\; 0$ and the existence of a Markov dilation for (T_t). We give applications to quantum metric spaces and show the equivalence of semigroup Hardy norms and martingale Hardy norms derived from the Markov dilation. In the limiting case we obtain a viable definition of BMO spaces for general semigroups of completely positive maps which can be used as an endpoint for interpolation. For torsion free ordered groups we construct a connection between Riesz transforms and the Hilbert transform induced by the order.

Introduction:
Riesz transforms provide important examples in classical harmonic analysis and have been studied extensively in the literature in many different aspects. The aim of this paper is to continue the work of P. A. Meyer, Bakry, Emery, Ledoux, Varopoulos and many others on probabilistic aspects of the theory of Riesz transforms, however in the noncommutative setting. The importance of analyzing semigroups of completely positive maps on von Neumann algebras has been impressively demonstrated by the recent work of Popa [Pop06], Peterson [Pel06], Popa and Ozawa [OP] and also occurs in the work of Shlyahktenko/Connes [CS05] on Betti numbers for von Neumann algebras. A common thread in this analysis is to adapt some differential geometric concepts in the setting of von Neumann algebras.

It was discovered by P.A. Meyer that the general theory of semigroups provides an appropriate framework to formulate Riesz transforms which relate the norm of different derivatives in the classical setting. To be more precise we consider a family (T_t) of contractive completely positive maps on a finite von Neumann algebra N with normal faithful trace τ such that

$$\tau(T_tx) \le \tau(x)$$

holds for positive x and $t > 0$. In the classical setting this is certainly satisfied for a semigroup of measure preserving positive maps on $N = L_\infty(\Omega, \Sigma, \mu)$. Then the maps T_t act on all L_p spaces $L_p(N, \tau)$ and in particular on the Hilbert space $L_2 = L_2(N, \tau)$. Let A be the negative generator of $T_t = e^{-tA}$. For nice elements the gradient form

$$2\Gamma(x,x) \;=\; A(x^*)y + x^*A(y) - A(x^*y)$$

is defined. In all our examples we will assume that the T_t's are selfadjoint, i.e. $\tau(T_txy) = \tau(xT_ty)$ and then A is indeed a positive (unbounded) operator. Under this circumstances we

The first author is partially supported by the National Science Foundation Foundation DMS 05-56120. The second author was partially supported by a Young Investigator Award of the N.S.F supported summer workshop in Texas A&M university 2007.

can formulate P.A. Meyer's problem: Is it true that

$$\|\Gamma(x,x)^{1/2}\|_p \sim_{c(p)} \|A^{1/2}x\|_p \tag{0.1}$$

holds for all reasonable elements x?

Let us illustrate this question by considering the Laplace operator $A(f) = -\Delta(f)$, where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$. Then it is easily verified that

$$\Gamma(f,h) \;=\; \sum_{i=1}^n \overline{\frac{\partial f}{\partial x_i}} \frac{\partial f}{\partial x_i} \;=\; (\nabla f, \nabla h)\;.$$

In this context P.A. Meyer's inequality reads as follows

$$\||\nabla f|\|_p \sim_{c(p)} \||\Delta|^{1/2}(f)\|_p\;.$$

In dimension $n=1$ this follows easily from the continuity of the Hilbert transform. In higher dimension this are the first examples of singular integrals and we refer to Stein's work [Ste70b, Ste70a] for credentials and further information. Indeed, P.A. Meyer was strongly motivated to provide a probabilistic approach to Stein's work on Riesz transforms and succeeded in showing his estimate for the Ornstein-Uhlenbeck semigroup, even in the infinite dimensional case [Mey76a, Mey76b]. In some sense Bakry [Bak85a, Bak85b, Bak87, Bak90, Bak94, Bak94] continued Meyer's line of research and showed (0.1) for many diffusion semigroups satisfying the $\Gamma^2 \;\geq\; 0$ condition. In the context of semigroups given by the Laplace-Betrami operator on a manifold the positivity of Γ^2 is equivalent to the positivity of the Ricci curvature. We recall that

$$2\Gamma^2(x,y) \;=\; \Gamma(Ax,y) + \Gamma(x,Ay) - A\Gamma(x,y)\;.$$

More generally the higher order gradients are defined as

$$2\Gamma^{k+1}(x,y) \;=\; \Gamma^k(Ax,y) + \Gamma^k(x,Ay) - A\Gamma^k(x,y)\;.$$

The Bochner identities for manifolds show that

$$\Gamma^2(f,f) \;=\; \mathrm{Ric}(df,df) + \|\nabla df\|_{HS}^2\;.$$

Here ∇ is the second covariant derivative and HS stands for the Hilbert Schmidt of the corresponding matrix of second derivatives.

In the noncommutative setting the notion of diffusion process is not (yet) well-defined. It is however clear that Meyer's approach requires the semigroup to have a *Markov dilation*. This means that there exists a family of homomorphisms $\pi_s : N \to \tilde{N}$ such that $\pi_s(N)$ is contained in a filtration $\tilde{N}_t$ with conditional expectation E_t such that

$$E_t(\pi_s(x)) \;=\; \pi_t(T_{s-t}x)$$

holds for $t < s$. Our main result is one half of P.A. Meyer's inequality for $p \;\geq\; 2$.

Theorem 0.1. *Let (T_t) be a semigroup of completely positive selfadjoint maps with Markov dilation and $\Gamma^2 \;\geq\; 0$. Let $2 \leq p < \infty$ then*

$$\|A^{1/2}x\|_p \;\leq\; c(p)\max\{\|\Gamma(x,x)^{\frac{1}{2}}\|_p, \|\Gamma(x^*,x^*)^{1/2}\|_p\}\;.$$

The assumptions of Theorem 0.1 are satisfied for Fourier multipliers on discrete groups, sometimes also called Herz-Schur multipliers. Indeed, let G be a discrete group and $VN(G)$ the group von Neumann algebra given by the left regular representation $\lambda : G \to B(\ell_2(G))$, $\lambda(g)\delta_h = \delta_{gh}$. It is well known that $\tau(\sum_g a_g\lambda(g)) = a_e$ extends to a normal faithful trace on $VN(G)$. A Fourier multiplier is given by a semigroup (ϕ_t) of positive definite functions and the normal extensions of

$$T_t(\lambda(g)) \;=\; \phi_t(g)\lambda(g)\;.$$

Following the recent work of Ricard, we know that T_t has a Markov dilation provided that $\phi_t(1) = 1$, ϕ_t is real valued and $\phi_t(g) = \phi_t(g^{-1})$, i.e. in the selfadjoint case. Let us note that then, according to Schoenberg's theorem, there exists a conditionally negative function $\psi : G \to [0, \infty)$. We obtain an immediate application to quantum metric spaces. Recall that a triple $(\mathcal{A}, B, \| \, \|_{Lip})$ is a quantum metric space, if $\mathcal{A} \subset B$ is a dense, not necessarily closed, sub*-algebra of a C^*-algebra B and $||| \, |||_{Lip}$ is a norm on $\mathcal{A}$ such that

$$|||ab|||_{Lip} \ \le \ |||a|||_{Lip} \, \|b\|_B + \|a\|_B \, |||b|||_{Lip}$$

and that

$$d_{Lip}(\phi, \psi) \ = \ \sup_{|||a|||_{Lip} \le 1} |\phi(a) - \phi(b)|$$

induces the weak* topology on the state space of B. Equivalently, the natural quotient map $\iota : (\mathcal{A}, ||| \, |||_{Lip}) \to B/\mathbb{C}1$ is a compact operator (see [OR05]). Theorem 0.1 allows us to show the compactness condition for discrete groups with rapid decay. Let us recall that a finitely generated group has rapid decay if

$$\|x\|_\infty \ \le \ C(s) k^s \|x\|_2$$

holds for some s and every $x = \sum_{|g|=k} a_g \alpha(g)$ supported on the words of length k. This notion is independent of the choice of the generators.

Corollary 0.2. *Let G be a finitely generated discrete group with rapid decay and $\psi : G \to \mathbb{C}$ be a conditionally negative function such that $\psi(1) = 0$, $\psi(g) = \psi(g^{-1})$, and*

$$\inf_{|g|=k} |\psi(g)| \ \ge \ c_\alpha k^\alpha$$

holds for some $\alpha > 0$. Then $\mathbb{C}[G]$ equipped with the norm

$$|||x||| \ = \ \|\Gamma_A(x, x)\|^{\frac{1}{2}}$$

defines a quantum metric space $(\mathbb{C}[G], C_{red}(G), ||| \, |||)$.

In view of the examples from Riemann manifolds it is clear that $||| \, |||$ is the "correct" Lipschitz norm.

Compared to Bakry's work in the commutative case the assumption "diffusion semigroup" is deleted in our Theorem 0.1 (but we have to keep the assumption of a nice algebra of invariant functions). In section 2 we provide a first proof of Theorem 0.1 requiring some extra regularity assumptions (easily verified for Fourier multipliers). In section 3 we follow Meyer/Bakry's footsteps. Unfortunately, this also requires to develop the theory of H_p spaces for noncommutative continuous filtrations, see [JK]. A key ingredient in Meyer's martingale approach is the use of a stopped brownian motion P_{B_s} as indices for the associated Poisson semigroup. More explicitly, let π_t be a Markov dilation, $a > 0$, B_t a Brownian motion with $B_0 = a$ almost everywhere and $\mathbf{t}_a(\omega) = \inf\{t : B_t(\omega) = 0\}$ the hitting time for the boundary. Then Meyer's approach consists in investigating the martingale

$$\rho_a(x) \ = \ \pi_{\mathbf{t}_a}(x) \ .$$

The H_p^c norm of the resulting martingale decomposes in a time and a space component. Without going into details let us mention that we are able to "compare" the martingale H_p^c norm in Meyer's model and show that they there are (almost) equivalent to the Hardy norms for semigroups investigated in the joint work C. Le Merdy and Q. Xu, see [JLMX06]. This explicit connection between martingale and semigroup H_p norms seems to be new and extends to the H_p norms given by the reversed Markov filtration.

Moreover, we are able to define spaces of bounded mean oscillation for quite general semigroups as follows

$$\|x\|_{BMO_c(T)} \ = \ \sup \|T_t|x|^2 - |T_t x|^2\|_\infty^{1/2} \ .$$

This norm is closely related to Garsia's BMO norm for the Poisson semigroup on the circle. The space BMO is defined as the completion of N with respect to the norm $\|x\|_{BMO} = \max\{\|x\|_{BMO_c}, \|x^*\|_{BMO_c}\}$. The good news is that the BMO space serves as an endpoint for interpolation.

Theorem 0.3. *Let (T_t) be a semigroup with Markov dilation and $\Gamma^2 \geq 0$. Then*

$$[BMO, L_p(N)]_{\frac{1}{q}} \; = \; L_{pq}(N) \, .$$

For the associated Poisson semigroup $P_t = e^{-tA^{1/2}}$ the connection between Garsia's BMO norm and Meyer's martingale approach is very explicit

$$\|x\|_{BMO_c(P)} \; = \; \|\rho_a(x)\|_{BMO_c}$$

holds for every a. This allows us to reduce the interpolation theorem to previous results on martingales. It is also a good indication that we have found an appropriate BMO spaces for semigroups. In section 4 we confirm this observation by showing that many natural candidates for BMO norms are equivalent. Section 6 is devoted to applications of Theorem 0.1 and Theorem 0.3. The applications towards quantum metric spaces are prepared in Section 5. Our last applications concerns torsion free ordered groups which admit a filtration of normal divisors $G = G_0 \supset G_1 \supset G_2 \supset \cdots$ such that

$$\bigcap_k G_k \; = \; \{e\} \quad , \quad G_k/G_{k+1} \; = \; \mathbb{Z} \, .$$

This holds for example for free groups in n generators. Using the extension $id \otimes P_t^{\mathbb{Z}}$ of the classical Poisson group on $G \times \mathbb{Z}$ we are able to reduce the boundedness of the Hilbert transform for ordered groups to estimates for Riesz transforms associated with P_t. This gives a link between the H_p-theory related to sub-diagonal von Neumann algebras and the H_p-theory for semigroups, see Section 6 for more details. In the text the absolute constant $c(p)$ may differ from line to line.

1. Preliminaries and notation

We will use standard notation in the theory of operator algebras which can be found in [Tak79, Tak03a, Tak03b], [KR97a, KR97b] or [SZ79, Str81]. Modular theory does not play an important role in this paper, because in most of our applications we are interested in von Neumann algebras with a finite trace. At any rate, using the Haagerup reduction method (see [HJX]) it usually suffices to consider the finite case. As a standard reference for noncommutative L_p-spaces we refer to [PX03] and the references therein. For basic properties of the space of τ-measurable operators and noncommutative integration we refer to [Nel74]. We will also use operator space terminology, in particular the notion of completely bounded maps, see the books by Effros-Ruan [ER00], Pisier [Pis03] or Paulsen [Pau02]. We allow for a slight deviation in the notion of *completely bounded* maps $T : X \to Y$, where $X \subset L_p(N)$, $Y \subset L_p(M)$ are subspaces of a noncommutative L_p space. Indeed, we use

$$\|T\|_{cb} \; = \; \sup_{\mathcal{M}} \|id \otimes T : L_p(\mathcal{M}; X) \to L_p(\mathcal{M}; Y)$$

where the supremum is taken over all von Neumann algebras $\mathcal{M}$ and the space

$$L_p(\mathcal{M}; X) \subset L_p(\mathcal{M} \otimes N)$$

is the completion of the tensor product $L_p(\mathcal{M}) \otimes X$ with respect to the induced norm. In the usual definition of the cb-norm, the supremum is only taken over $\mathcal{M} = K(\ell_2)$, the compact operators on ℓ_2. If Connes' embedding conjecture were true the two norms coincide. Our policy in general is to prove the estimates with respect to the stronger norm. Indeed, as so often in

martingale theory these estimates are automatic, i.e. follow because T and $id \otimes T$ satisfy the same assumptions.

We will frequently use standard tools from noncommutative probability, in particular Doob's inequality

$$\|\sup_n{}^+ E_n(x)\|_p \;\le\; d_p\|x\|_p$$

for $1 < p \le \infty$ and the dual Doob inequality

$$\|\sum_n E_n(x_n)\|_p \;\le\; c_p\|\sum_n x_n\|_p$$

which holds for $1 \le p < \infty$ and positive x_n with the constant $c_p = d_{p'}$. Here (E_n) is a sequence of normal conditional expectations onto an increasing or decreasing sequence of von Neumann subalgebras (N_n) of a given von Neumann N. In the finite setting these conditional expectations are unique and hence commute, i.e. $E_nE_m = E_mE_n = E_{\min(n,m)}$ (increasing filtration) or $E_nE_m = E_mE_n = E_{\max(n,m)}$ (decreasing filtration). We will always assume this commutation relation. The notation $\sup_n{}^+$ is taken from [Jun02] and [JX03]. In the noncommutative setting the pointwise supremum can not be defined directly. However, for selfadjoint operators x_n we have an order analogue

$$\|\sup_n{}^+ x_n\|_p \;=\; \inf_{-a\le x_n\le a}\|a\|_p \,.$$

In full generality we use Pisier's definition

$$\|\sup_n{}^+ x_n\|_p \;=\; \inf_{x_n = ay_nb}\|a\|_{2p}\sup_n\|y_n\|_\infty\|b\|_{2p} \,.$$

The same definition holds for arbitrary index sets. We also need some basic facts about H_p-spaces for martingales. Let us recall some definitions from [JX08]. As usual the martingale difference are denoted by $d_kx = E_kx - E_{k-1}x$.

$$\|x\|_{H_p^c} = \|(\sum_k d_kx^*d_kx)^{\frac12}\|_p \,, \|x\|_{h_p^c} \;=\; \|(\sum_k E_{k-1}(d_kx^*d_kx))^{\frac12}\|_p \,, \|x\|_{h_p^d} \;=\; (\sum_k \|d_kx\|_p^p)^{\frac1p} \,.$$

The row versions are given by $\|x\|_{H_p^r} = \|x^*\|_{H_p^c}$, $\|x\|_{h_p^r} = \|x^*\|_{h_p^c}$. In the noncommutative theory the definition of the H_p-spaces is as follows

$$H_p \;=\; \begin{cases} H_p^c \cap H_p^r & \text{if } p \;\ge\; 2 \\ H_p^c + H_p^r & \text{if } 1 \le p \le 2 \end{cases} .$$

The Burkholder Gundy inequality reads as

$$L_p(N) \;=\; H_p \quad 1 < p < \infty \,. \tag{1.1}$$

The Burkholder inequalities can be formulated as

$$L_p(N) \;=\; \begin{cases} h_p^c \cap h_p^r \cap h_p^d & p \;\ge\; 2 \,, \\ h_p^c + h_p^d + h_p^r & 1 < p \le 2 \,. \end{cases} \tag{1.2}$$

All the equalities hold with equivalent norms. Since this will be needed in the paper we want to show $H_p = h_p^d + h_p^c$ for $1 \le p \le 2$. This requires us to use the dual norms

$$\|x\|_{L_p^cMO} \;=\; \|\sup_n{}^+ E_n(|x - E_{n-1}(x)|^2)\|_{p/2}^{1/2} \,, \; \|x\|_{L_p^cmo} \;=\; \|\sup_n{}^+ E_n(|x - E_n(x)|^2)\|_{p/2}^{1/2} \,.$$

Extending the Fefferman Stein duality $(\overline{H_1^c})^* = BMO_c$ from [PX97] it was shown in [JX03] that

$$\overline{H_p^c}^* \;=\; L_{p'}^cMO \quad 1 \le p < 2 \,.$$

Here $\overline{X}^*$ refers to the anti-linear duality $\langle x, y\rangle = tr(x^*y)$. The following observation is probably of independent interest.

Lemma 1.1. *Let $1 \le p < 2$ and $(N_k)_{k\,\ge\,1}$ be a martingale filtration*

i) *Let $L_p^{c,cond} = \{(x_k)_k \;:\; x_k \in N_k\} \subset L_p(N;\ell_2^c)$. Then the antilinear dual of $L_p^{c,cond}$ is isomorphic to the space $L_{p'}^{c,cond}MO$ of sequences (x_k) with $x_k \in L_{p'}(N_k)$ such that*

$$\|(x_k)\|_{L_{p'}^{c,cond}MO} \;=\; \|\sup_n E_n(\sum_{k\,\ge\,n} x_k^*x_k)\|_{p'/2}^{1/2} \,.$$

ii) *Let $h_p^{c,\circ}$ be the subspace of h_p^c of elements with $d_1 = 0$. The anti-linear dual of $h_p^{c,\circ}$ is $L_{p'}^c mo$.*

iii) *$H_p^c \;=\; h_p^d + h_p^c$.*

Proof. In [Jun02] it is shown that

$$|\sum_k tr(x_k^*y_k)| \;\le\; \sqrt{2}\,\|(\sum_k E_k(x_k^*x_k))^{\frac{1}{2}}\|_p\,\|\sup_n E_n(\sum_{k\,\ge\,n} y_k^*y_k)\|_{p'/2}^{\frac{1}{2}} \,. \tag{1.3}$$

In particular, we have a continuous inclusion $L_{p'}^{c,cond}MO \subset (\overline{L_p^{c,cond}})^*$. For the converse we note that $L_p^{c,cond}$ is a subspace of $L_p(N,\ell_2^c)$. Hence a linear functional $f : \overline{L_p^{c,cond}} \to \mathbb{C}$ of norm one is given by a sequence $(z_k) \subset L_{p'}(N,\ell_2^c)$ such that

$$f(x) \;=\; \sum_k tr(x_k^*z_k) \,.$$

We define $y_k = E_k(z_k)$ and deduce from Doob's inequality for $p'/2 > 1$ that

$$\begin{aligned}\|\sup_n E_n(\sum_{k\,\ge\,n} y_k^*y_k)\|_{p'/2} &\le \|\sup_n E_n(\sum_{k\,\ge\,n} E_k(z_k^*z_k))\|_{p'/2} \;\le\; \|\sup_n E_n(\sum_k z_k^*z_k)\|_{p'/2}\\ &\le d_{p'/2}\|\sum_k z_k^*z_k\|_{p'/2} \,.\end{aligned}$$

For the proof of ii) we assume $d_1(x) = 0$ or $d_1(y) = 0$ and note that according to (1.3) we have

$$\begin{aligned}|tr(x^*y)| &= |\sum_{k\,\ge\,2} tr(d_k(x)^*d_k(y))|\\ &\le\; \sqrt{2}\,\|(\sum_{k\,\ge\,2} E_{k-1}(d_k(x)^*d_k(x)))^{\frac{1}{2}}\|_p\,\|\sup_{n\,\ge\,1} E_n(\sum_{k-1\,\ge\,n} d_k(y)^*d_k(y))\|_{p'/2}^{\frac{1}{2}} \,.\end{aligned}$$

Let us recall that $L_p^c mo$ consists of martingales with $d_1(y) = 0$. It has been proved in [Jun02] that there are linear maps $u_k : N \to C\bar{\otimes}(N_k)$ (the space of weakly converging columns with values in N_k) such that

$$u_k(x)^*u_k(x) \;=\; E_k(x^*x) \,.$$

Moreover, u_k is a N_k right module map with complemented range (see [Jun02] and for the non-separable case [JS05]). Then $u : h_p^{c,\circ} \to L_p^{c,cond}$ given by $u(x) \;=\; (u_{k-1}(d_k(x))_{k\,\ge\,2}$ is an isometric isomorphism. Hence an antilinear functional $f : \overline{h_p^{c,\circ}} \to \mathbb{C}$ is given by a sequence $z_k \in L_{p'}(N_{k-1},\ell_2^c)$ such that

$$f(u(x)) \;=\; \sum_k tr(u_{k-1}(d_k(x))^*z_k)$$

and $\|\sup_n E_n(\sum_{k-1\,\ge\,n} z_k^*z_k)\|_{p'/2} \le c_p\|f\|$. Since the range of u_{k-1} is complemented, we may use the projection and find y_k such that $u_{k-1}(y_k) = Pz_k$ and

$$\|\sup_n E_n(\sum_{k-1\,\ge\,n} y_k^*y_k)\|_{p'/2} \;=\; \|\sup_n E_n(\sum_{k-1\,\ge\,n} E_{k-1}(y_k^*y_k))\|_{p'/2}$$

$$= \| \sup_n E_n(\sum_{k-1 \, \ge \, n} u_{k-1}(y_k)^* u_{k-1}(y_k)) \|_{p'/2} \; \le \; \| \sup_n E_n(\sum_{k-1 \, \ge \, n} z_k^* z_k) \|_{p'/2} \; \le \; c_p \|f\| \, .$$

We define $y = \sum_k d_k(y_k)$. Using the triangle inequality and $E_n E_k = E_n$ we get that

$$\begin{aligned} &\| \sup_n E_n(\sum_{k-1 \, \ge \, n} d_k(y)^* d_k(y)) \|_{p'/2}^{\frac{1}{2}} \\ &\le \| \sup_n E_n(\sum_{k-1 \, \ge \, n} E_k(y_k)^* E_k(y)) \|_{p'/2}^{\frac{1}{2}} + \| \sup_n E_n(\sum_{k-1 \, \ge \, n} E_{k-1}(y_k)^* E_{k-1}(y)) \|_{p'/2}^{\frac{1}{2}} \\ &\le \; 2 \, \| \sup_n E_n(\sum_{k-1 \, \ge \, n} y_k^* y_k) \|_{p'/2} \; \le \; 2c_p \, \|f\| \, . \end{aligned}$$

For the proof of iii) we recall that $h_p^d + h_p^c \subset H_p^c$. We claim that the unit ball $B_{H_p^c}$ is contained in the norm closure of $\overline{C(B_{h_p^d} + B_{h_p^c})}^{\| \; \|_{H_p^c}}$ for some constant $C > 0$. If not there exists $x \in H_p^c$ and a continuous linear functional y such that $tr(y^* x) = 1$ and

$$|tr(y^* x')| \; \le \; \frac{1}{C} \quad \text{for all} \quad x' \in CB_{h_p^d} \cup CB_{h_p^c} \, .$$

We know that $y = \sum_n d_n$ satisfies

$$\begin{aligned} &\|y\|_{(\overline{H}_p^c)^*} \; \le \; \sqrt{2} \|y\|_{L_p^c MO}^2 \; \le \; \sqrt{2} \, \| \sup_n d_n^* d_n \|_{p'/2} + \sqrt{2} \, \| \sup_n E_n(\sum_{k>n} d_k^* d_k) \|_{p'/2} \\ &\le \sqrt{2} \left(\sum_n \|d_n\|_{p'}^{p'} \right)^{\frac{1}{p'}} + \sqrt{2} \| \sup_n E_n(\sum_{k>n} d_k^* d_k) \|_{p'/2} \\ &\le \; 2\sqrt{2} \, \|y\|_{\overline{h_p^d}^*} + \sqrt{2} c_p \|y\|_{(\overline{h_p^c})^*} \; \le \; \frac{\sqrt{2}(2 + c_p)}{C} \, . \end{aligned}$$

Since $\|x\|_{H_p^c} \le 1 = tr(y^* x)$ we reach a contradiction for $C > \sqrt{2}(2 + c_p)$. An approximation argument allows us to replace the norm closure of $C(B_{h_p^d} + B_{h_p^c})$ by $2C(B_{h_p^d} + B_{h_p^c})$. ■

Let us briefly prove the martingale inequalities for potentials in the noncommutative setting. We recall a classical martingale inequality from [Jun02] which is derived from (1.3). Let (N_k) be a (discrete) increasing filtration and $a_k \in N$ be positive elements. For $p \; \ge \; 2$ we have

$$\| \sum_k E_k(a_k) \|_{\frac{p}{2}} \; \le \; 2c_p^2 \, \| \sup_m \sum_{k \, \ge \, m} E_m(a_k) \|_{\frac{p}{2}} \, . \tag{1.4}$$

Here c_p is the constant in Stein's inequality. Let $(z_k)_{k \le n}$ be a finite submartingale, i.e.

$$E_k(z_{k+1}) \; \ge \; z_k \, .$$

The corresponding positive increasing part of z is defined as

$$a_k \; = \; \langle z \rangle_k \; = \; \sum_{j<k} E_j(z_{j+1} - z_j) \, .$$

Clearly, we obtain a martingale

$$m_k \; = \; z_k - a_k \, .$$

Indeed, $m_k - m_{k-1} = z_k - z_{k-1} - E_{k-1}(z_k - z_{k-1})$ is a martingale difference sequence. In the language of potentials, we have

$$E_j(z_{j+1} - z_j) \; = \; E_j(m_{j+1} - m_j) + E_j(a_{j+1} - a_j) \; = \; E_j(a_{j+1} - a_j)$$

Moreover, we note that $a_{j+1}-a_j$ is still positive. Hence for $p>1$, we deduce from (1.4) and Doob's inequality

$$\begin{aligned}\|a_n\|_p &= \|\sum_{j=1}^n E_j(z_{j+1}-z_j)\|_p = \|\sum_{j=1}^n E_j(a_{j+1}-a_j)\|_p \\ &\le 2c_{2p}^2 \|\sup_m{}^+ \sum_{m\,\le\, j\le n} E_m(a_{j+1}-a_j)\|_p \le 2c_{2p}^2\|\sup_m{}^+ E_m(z_n-z_m)\|_p \\ &\le 2c_{2p}^2\|\sup_m{}^+ E_m(z_n)\|_p + \|\sup_m z_m\|_p \le 2c_{2p}^2 d_p\|z_n\|_p + \|\sup_m z_m\|_p\,.\end{aligned}$$

If in addition $z_m \ge 0$ for all m, we may ignore the second term and obtain the following result.

Lemma 1.2. *Let $z_k = a_k+m_k$ be a positive submartingale with increasing part (a_k) and martingale part (m_k). Then*

$$\|a_n\|_p \le c(p)\,\|z_n\|_p$$

holds for $1<p<\infty$ and some universal constant $c(p)$.

The H_p theory for continuous filtrations $(N_t)_{t\,\ge\,0}\subset N$ has only been considered recently (see [JK]). We will always assume that $\bigcap_{s>t} N_s = N_t$. It is well-known that the theory of H_p-norms is closely related to stochastic integrals. However, given how nicely the H_p-theory translates in the noncommutative setting, we should not expect surprises. There are two candidates for the H_p^c-norm on a finite interval $[0,T]$

$$\|x\|_{H_p^c} = \lim_{\sigma,\mathcal{U}}\|\sum_{j=0}^{|\sigma|-1}|E_{s_{j+1}}x-E_{s_j}x|^2\|_{p/2}^{1/2} \quad , \quad \|x\|_{\hat{H}_p^c} = \|\lim_{\sigma,\mathcal{U}}\sum_{j=0}^{|\sigma|-1}|E_{s_j+1}x-E_{s_j}x|^2\|_{p/2}^{1/2}\,.$$

Here $\sigma=\{0=s_0,...,s_n=T\}$ is a partition $|\sigma|=n$ and $\mathcal{U}$ is an ultrafilter refining the natural order of inclusion on the set of all partitions. In the second term we take the weak*-limit (at least for $p\ge 2$). It was shown that in [JK] that the two norms are equivalent and, up to a constant c_p, independent of the choice of $\mathcal{U}$. The main tool in this argument is the observation that $H_p^c = h_p^d\cap h_p^c$ for $p\ge 2$, where

$$\|x\|_{h_p^d} = \lim_{\sigma,\mathcal{U}}(\sum_{j=0}^{n-1}\|E_{s_{j+1}}x-E_{s_j}x\|_p^p)^{\frac1p} \quad , \quad \|x\|_{h_p^c} = \lim_{\sigma,\mathcal{U}}\|\sum_{j=0}^{n-1}E_{s_{j-1}}|E_{s_{j+1}}x-E_{s_j}x|^2\|_{p/2}^{1/2}\,.$$

Again it is shown that the limit can be taken inside. This gives the norm $\|\cdot\|_{\hat{h}_p^c}$ and conditioned bracket

$$\langle x,x\rangle_T = \lim_{\sigma,\mathcal{U}}\sum_{j=0}^{n-1}E_{s_{j-1}}|E_{s_{j+1}}x-E_{s_j}x|^2$$

and

$$\|x\|_{h_p^c}\sim_{c(p)}\|\langle x,x\rangle_T\|_{p/2}^{1/2} = \|x\|_{\hat{h}_p^c}\,.$$

In the continuous context the Burkholder inequalities reads as follows

$$L_p(N) = \begin{cases}\hat{h}_p^c\cap\hat{h}_p^r\cap h_p^d & \text{if } p\ge 2\\ \hat{h}_p^c+\hat{h}_p^r+h_p^d & \text{if } 1<p\le 2\end{cases},$$

where $h_p^r,\hat{h}_p^r$ are the corresponding row spaces. The exact form of the Feffermann-Stein duality for $p=1$ is not yet explored. For $2\ge p>1$ we used the definition $\hat{h}_p^c=\overline{\hat{h}_{p'}^c}^{\,*}$ and refer to [JK] for more information. A martingale x is said to have *a.u. continuous path* if for every $T>0$,

every $\varepsilon > 0$ there exists a projection e with $\tau(1-e) < \varepsilon$ such that the function $f_e : [0,T] \to N$ given by

$$f_e(t) \;=\; x_t e \in N$$

is continuous. For a martingale with a.u. continuous path we have $\mathrm{var}_p(x) = \|x\|_{h_p^d} = 0$ for all $2 < p < \infty$. We recall from [JK] that the condition $\mathrm{var}_p(x) = 0$ implies that

$$\lim_{\sigma,\mathcal{U}} \|\sum_{j=0}^{n-1} |E_{s_j+1}x - E_{s_j}x|^2 - \sum_{j=0}^{n-1} E_{s_j}(|E_{s_j+1}x - E_{s_j}x|^2)\|_{p/2} \;=\; 0$$

for all $p > 2$ and the norm convergence of

$$L_{p/2} - \lim_{\sigma} \sum_{j=0}^{n-1} E_{s_j}(|E_{s_j+1}x - E_{s_j}x|^2) \;=\; \langle x, x\rangle_T \;.$$

This implies that for martingales with $\mathrm{var}_p(x) = 0$ the equality (see [JK])

$$\|x\|_{H_p^c([0,T])} \;=\; \|\langle x, x\rangle_T\|_{p/2}^{1/2}$$

holds without constants. Also we have

$$\lim_{\sigma,\mathcal{U}} \|\sup_j{}^+ E_{s_j}(|E_T x - E_{s_{j-1}}x|^2)\|_{p/2} \;=\; \|\sup_s{}^+ E_s(\langle x, x\rangle_T - \langle x, x\rangle_s)\|_{p/2} \;. \tag{1.5}$$

The correct definition of the norm in $L_p^c mo$ for continuous filtration is

$$\|x\|_{L_p^c mo} \;=\; \sup_T \|\sup_s{}^+ E_s(\langle x, x\rangle_T - \langle x, x\rangle_s)\|_{p/2}^{1/2}$$

for $2 \le p \le \infty$.

It is shown in [JX07] that the ergodic averages $M_t x = \frac{1}{t}\int_0^t T_s x ds$ satisfy a maximal inequality

$$\|\sup_t M_t(x)\|_p \;\le\; d_p \|x\|_p \quad 1 < p \le \infty \tag{1.6}$$

and the dual inequality

$$\|\sum_k M_{t_k}(x_k)\|_p \;\le\; c_p \|\sum_k x_k\|_p \quad 1 \le p < \infty \tag{1.7}$$

for positive x_k. We will make extensive use of the Poisson semigroup

$$P_s \;=\; \frac{1}{2\sqrt{\pi}} \int_0^\infty s e^{-\frac{s^2}{4u}} u^{-\frac{3}{2}} T_u du. \tag{1.8}$$

It has been shown in [JX07] that P_s is a positive average of the M_t and hence (1.6) and (1.7) hold for P_t instead of M_t. If we assume in addition that the T_t's are selfadjoint, i.e.

$$\tau(T_t x y) \;=\; \tau(x T_t y) \tag{1.9}$$

holds for all $x, y \in N$, then (1.6) and (1.7) holds for the T_t's instead of the M_t, see [JX08]. We will impose a *standard assumption*, namely the existence of weak*-dense, not necessarily closed *-algebra $\mathcal{A} \subset N$ such that

$$T_t(\mathcal{A}) \subset \mathcal{A} \quad \text{and} \quad A(\mathcal{A}) \subset \mathcal{A}$$

holds for $t > 0$ and the negative generator A of the semigroup $T_t = e^{-tA}$. Whenever we talk about P_t we will also assume that $P_t(\mathcal{A}) \subset \mathcal{A}$. This assumption is satisfied for our main examples, Fourier multipliers, but somewhat problematic for certain manifolds. We are convinced that with some extra effort our results still hold for these manifolds and most of the interesting examples, but the arguments are far more transparent with this additional assumption. We will make crucial use of the following inequality from [Mei08].

Proposition 1.3. *Let $x \in \mathcal{A}$ be positive and $0 < t < s$. Then*
$$P_s x \;\le\; \frac{s}{t}\, P_t x \;.$$

Proof. We use (1.8) and $e^{-\frac{s^2}{4u}} \le e^{-\frac{t^2}{4u}}$ for all u. This yields the assertion
$$\frac{P_s x}{s} = \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{s^2}{4u}} u^{-\frac{3}{2}} T_u(x)du \;\le\; \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u(x)du \;=\; \frac{P_t x}{t} \;. \qquad \blacksquare$$

We will use $H_p(T), H_p(P)$ to denote the Hardy spaces associated with semigroup (T_t) and subordinated Poisson semigroup (P_t) respectively. See [JLMX06] for more details.

2. Reversed martingale filtration

We will assume that (T_t) is a semigroup of completely positive maps on a tracial von Neumann algebra N. A Markov dilation for T_t is given by a family $\pi_s : N \to \mathcal{M}$ of trace preserving *-homomorphism with the following properties

i) Let $M_{s]}$ be the von Neumann algebra generated by the $\pi_v(x)$, $x \in N$, $v \le s$. Then
$$E_{s]}(\pi_t(x)) \;=\; \pi_s(T_{t-s}x)$$
holds for $s < t$ and $x \in N$.

ii) Let $M_{[s}$ be the von Neumann algebra generated by the $\pi_v(x)$, $x \in N$, $v \ge s$. Then
$$E_{[s}(\pi_t(x)) \;=\; \pi_s(T_{s-t}x)$$
holds for $t < s$ and $x \in N$.

This definition is adapted for selfadjoint (T_t). We recall that $\Gamma^2 \ge 0$ is equivalent to
$$\Gamma(T_t x, T_t x) \;\le\; T_t \Gamma(x,x) \;.$$
Whenever we invoke the associated Poisson semigroup $P_t x = e^{-tA^{1/2}}x$ we assume in addition that $P_t(\mathcal{A}) \subset \mathcal{A}$. It is easy to see that $\Gamma^2 \ge 0$ also implies
$$\Gamma(P_t x, P_t x) \;\le\; P_t \Gamma(x,x) \;.$$

Lemma 2.1. *Let (π_s) be a Markov dilation.*

i) *Then $m(x)$ defined by*
$$m_s(x) \;=\; \pi_s(T_s x)$$
is a martingale with respect to the reversed filtration $M_{[s}$.

ii) *Assume $\Gamma^2 \ge 0$ or $\Gamma(T_r x, T_r x)$ is uniformly bounded in L_2 for $x \in \mathcal{A}$. Then for $x \in N$ the martingale $m_s(x)$ has continuous path. If moreover, $f(r,s) = T_s\Gamma(T_r x, T_r x)$ is continuous in L_2, or $\Gamma^3 \ge 0$, or $\Gamma^2(T_r x, T_r x)$ is locally bounded in L_2, then*
$$\langle m(x), m(y)\rangle \;=\; 2\int_0^\infty \pi_t(\Gamma(T_t x, T_t x))dt \;.$$

Proof. The first assertion is obvious, because $E_{[s}(\pi_0(x)) = \pi_s(T_s x)$. For the second we assume that $x \in \mathcal{A}$ be selfadjoint. Let $s < t$. Then we deduce from the Cauchy Schwarz inequality in the form $|\tau(abab)| \le \tau(abba)$ that
$$\begin{aligned}
&\|m_s(x) - m_t(x)\|_4^4 \;=\; \|\pi_s(|T_s x|^2) + \pi_t(|T_t x|^2) - \pi_s(T_s x)\pi_t(T_t x) - \pi_t(T_t x)\pi_s(T_s x)\|_2^2 \\
&= \tau((\pi_s(|T_s x|^2) + \pi_t(|T_t x|^2))^2) + \tau((\pi_s(T_s x)\pi_t(T_t x) + \pi_t(T_t x)\pi_s(T_s x))^2) \\
&\quad - 2\tau((\pi_s(|T_s x|^2) + \pi_t(|T_t x|^2))(\pi_s(T_s x)\pi_t(T_t x) + \pi_t(T_t x)\pi_s(T_s x)) \\
&= \tau((T_s x)^4) + \tau((T_t x)^4) + 2\tau(\pi_s((T_s x)^2)\pi_t((T_t x)^2)) \\
&\quad + 2\tau(\pi_s(T_s x)\pi_t(T_t x)\pi_s(T_s x)\pi_t(T_t x)) + 2\tau(\pi_s((T_s x)^2)\pi_t((T_t x)^2))
\end{aligned}$$

$$
\begin{aligned}
&\quad - 2\Big(\tau(\pi_s((T_sx)^3)\pi_t(T_tx)) + \tau(\pi_t((T_tx)^3)\pi_s(T_sx))\Big)\\
&\le \tau((T_sx)^4) + \tau((T_tx)^4) + 6\tau(\pi_s((T_sx)^2)\pi_t((T_tx)^2))\\
&\quad - 4\big(\tau(\pi_s((T_sx)^3)\pi_s(T_{t-s}T_tx)) + \tau((T_tx)^4)\big)\\
&= \tau((T_sx)^4) + 6\tau(T_{t-s}(T_sx)^2(T_tx)^2) - 4\tau((T_sx)^3T_{2t-s}x) - 3\tau((T_tx)^4)\\
&= 3\Big(\tau((T_tx)^4) - \tau((T_sx)^4))\Big) + 6\Big(\tau(T_{t-s}(T_sx)^2(T_tx)^2) - \tau((T_tx)^4)\Big)\\
&\quad - 4\Big(\tau((T_sx)^3T_{2t-s}x) - \tau((T_sx)^4)\Big) .
\end{aligned}
$$

Now, it suffices to consider the terms separately. Let $d = T_tx - T_sx$. Then we use $|\tau(abab)| \le \tau(abba)$ again and find

$$
\begin{aligned}
|\tau((T_sx+d)^4 - (T_sx)^4)| &\le 4|\tau(d^3T_s(x)) + 4|\tau(dT_s(x)^3| + 2|\tau(dT_sxdT_sx)| + 4\tau(d^2(T_sx)^2)\\
&\le 4|\tau(d^3T_sx)| + 4|\tau(dT_s(x)^3)| + 6\tau(d^2(T_sx)^2)\\
&\le 12\|d\|_2 \max\{\|(T_tx-T_sx)^2T_sx\|_2, \|(T_sx)^3\|_2, \|(T_tx-T_sx)(T_sx)\|_2\} .
\end{aligned}
$$

For x in the domain of A, we know that $f(t) = T_tx$ is differentiable and hence

$$\|T_tx - T_sx\|_2 \;=\; \|\int_s^t T_rAx\|_2 \;\le\; (t-s)\|Ax\|_2 \tag{2.1}$$

Equation (2.1) also allows us to estimate the last term. Indeed, by Cauchy-Schwarz

$$|\tau((T_sx)^3T_{2t-s}x) - \tau((T_sx)^4)| \;\le\; 2(t-s)\|AT_sx\|_2\|T_sx\|_6^3 \;\le\; 2(t-s)\|Ax\|_2\|x\|_N^3 .$$

For the middle term we consider the function

$$f(r) \;=\; T_{t-r}(T_rx^*T_ry) .$$

Due to our assumption this function is differentiable and

$$f'(r) \;=\; T_{t-r}A(T_rx^*T_ry) - T_{t-r}(AT_rx^*T_ry) - T_{t-r}(T_rx^*AT_ry) \;=\; -2T_{t-r}\Gamma(T_rx, T_ry) .$$

This implies

$$T_{t-s}(|T_sx|^2) - |T_tx|^2 \;=\; \int_s^t 2T_{t-r}\Gamma(T_rx, T_rx)dr . \tag{2.2}$$

If $\Gamma^2 \;\ge\; 0$ we have

$$\|T_{t-s}(|T_sx|^2) - |T_tx|^2\|_2 \;\le\; 2(t-s)\|\Gamma(x,x)\| .$$

A similar estimate holds if just assume $\sup_r \|\Gamma(T_rx, T_rx)\|_2 \;\le\; C$, because T_{t-r} is a contraction on $L_2(N)$. This implies

$$\|m_t - m_s\|_4^4 \;\le\; (t-s)(40\;\|x\|_N^3\|Ax\|_2 + 12\;\|\Gamma(x,x)\|\|x\|_N^2)$$

for all $x \in \mathcal{A}$. The noncommutative version of Kolmogorov's theorem is proved in [GŁ85]. Thus $m_t(x)$ has continuous path. Due to Doob's inequality the class of martingales with continuous path is closed in $L_p(N)$. Since $\mathcal{A}$ is assumed to be weakly dense and hence norm dense in $L_p(N)$, we deduce the assertion for all $x \in N$. For the last formula we observe for $x \in \mathcal{A}$ that

$$E_{[t}(|\pi_s(T_sx) - \pi_t(T_tx)|^2) = E_{[t}(\pi_s(T_s|x|^2)) - \pi_t(|T_tx|^2) \;=\; \pi_t(T_{t-s}|T_sx|^2) - |T_tx|^2) .$$

We deduce from (2.2) that

$$E_{[t}(\pi_s(T_s|x|^2) - \pi_t(|T_tx|^2)) \;=\; \pi_t(\int_s^t 2T_{t-r}\Gamma(T_rx, T_rx)dr) \;=\; E_{[t}\int_s^t \pi_r(2\Gamma(T_rx, T_rx))dr . \tag{2.3}$$

This implies for the limit

$$\langle m(x),m(x)\rangle_s - \langle m(x),m(x)\rangle_t \;=\; \lim_{|\pi|\to 0}\sum_j E_{[s_{j+1}}(|\pi_{s_j}(T_{s_j}) - \pi_{s_{j+1}}(T_{s_{j+1}})|^2)$$
$$= \lim_{|\pi|\to 0}\sum_j E_{[s_{j+1}}\int_{s_j}^{s_{j+1}} 2\pi_r(\Gamma(T_rx,T_rx))dr \;=\; \int_s^t 2\pi_r(\Gamma(T_rx,T_rx))dr\ .$$

Here the limit is taken over refining partitions. Since $m(x)$ has continuous path, we know that the brackets coincide. In order to remove the extra term $E_{s_{j+1}}$ from the integral in the last inequality, we first note that for $r<s$

$$\|\pi_r(a)-E_s\pi_r(a)\|_2^2 = \|\pi_r(a)-\pi_s(T_{s-r}a)\|_2^2 \;=\; \tau(a^2-(T_{s-r}a)^2)$$
$$\le \|T_{s-r}a-a\|_2(\|a\|_2+\|T_{s-r}a\|_2) \;\le\; |s-r|\ \|Aa\|_2\ (\|a\|_2+\|T_{s-r}a\|_2)\ . \tag{2.4}$$

Applying this to $a=\Gamma(T_rx,T_rx)$, we are done assuming the continuity of $f(r,s)=T_s\Gamma(T_rx,T_rx)$ in L_2. In fact, we deduce from the Cauchy Schwarz inequality that

$$\begin{aligned}\|A\Gamma(T_rx,T_rx)\|_2 &\le 2\ \|\Gamma^2(T_rx,T_rx)\|_2 + \|\Gamma(AT_rx,T_rx)\|_2 + \|\Gamma(T_rx,AT_rx)\|_2\\ &\le 2\ \|\Gamma^2(T_rx,T_rx)\|_2 + 2\ \|\Gamma(T_rAx,T_rAx)\|_2^{\frac12}\|\Gamma(T_rx,T_rx)\|_2^{\frac12}\\ &\le 2\ \|\Gamma^2(T_rx,T_rx)\|_2 + 2\ \|\Gamma(Ax,Ax)\|_2^{\frac12}\|\Gamma(x,x)\|_2^{\frac12}\ .\end{aligned}$$

If $\Gamma^3 \ \ge\ 0$, we have $\|\Gamma^2(T_rx,T_rx)\| \le \|\Gamma^3(x,x)\|_2$ and $\Gamma^3(x,x)\in N$ for $x\in\mathcal{A}$. Assuming only the boundedness of $\Gamma^2(T_rx,T_rx)$, still allows us to obtain the assertion by refining the partition. ■

Remark 2.2. Under the condition ii) we see moreover that

$$\langle m(x),m(x)\rangle_s - \langle m(x),m(x)\rangle_t \;\le\; 2\,|s-t|\ \|\Gamma(x,x)\|_\infty$$

holds for $s<t$. By approximation this implies that the bracket is absolutely continuous with respect to the Lebesgue measure. In the commutative case this is enough to show that the martingale can be obtained as a stochastic integral against the brownian motion. We refer to [JK] for similar applications of this regularity condition.

Proposition 2.3. *Let $2\le p<\infty$ and $\Gamma^2\ \ge\ 0$. Let $x\in\mathcal{N}$ be a selfadjoint mean 0 element. Then*

$$c_p^{-1}\|x\|_p \;\le\; \|(\int_0^\infty \Gamma(T_tx,T_tx)dt)^{\frac12}\|_p \;\le\; c_p\|x\|_p\ .$$

Moreover, for every $x\in N$ there exists a martingale $m^2(x)$ such that

$$\|m^2(x)\|_{H_p^c} \;\le\; c(p)\|\int_0^\infty \Gamma(T_sx,T_sx)ds\|_{\frac{p}{2}}^{\frac12} \quad and \quad \tau(\pi_0(y)^*m^2(x)) \;=\; \frac13\tau(y^*(I-\mathrm{Pr})x)\ .$$

Here Pr is the projection onto the kernel of A. If in addition $f(r,s)=T_r\Gamma(T_sx,T_sx)$ is L_2-continuous for $x\in\mathcal{A}$, then

$$\langle m^2(x),m^2(x)\rangle \;=\; \int_0^\infty \pi_s(\Gamma(T_{2s}x,T_{2s}x))ds\ .$$

Proof. Let s_0 be fixed and $\sigma=\{0,...,s_0\}$ a partition. We define the martingale differences

$$d_j \;=\; (\pi_{s_j}(T_{s_j+s_{j+1}}x) - \pi_{s_{j+1}}(T_{2s_{j+1}}))\ .$$

Indeed, $E_{s_{j+1}}\pi_{s_j}(T_{s_j+s_{j+1}}x) = \pi_{s_{j+1}}(T_{2s_{j+1}}x)$ shows that

$$m_\sigma \;=\; \sum_{j=1}^{s_0} d_j \; .$$

is a martingale with respect to the discrete filtration $(N_{[s_j})$. Following Lemma 2.1 (in particular (2.3)), we may also calculate the conditioned bracket

$$\sum_j E_{[s_{j+1}}(|d_j|^2) \;=\; \sum_j E_{s_{j+1}} \int_{s_j}^{s_{j+1}} \pi_r(\Gamma(T_{r+s_{j+1}}x, T_{r+s_{j+1}}x))dr \; .$$

On the other hand, we deduce again from Lemma (2.1) that

$$\begin{aligned} \|d_j\|_4^4 \;&=\; \|m_{s_j}(T_{s_{j+1}}x) - m_{s_{j+1}}(T_{s_{j+1}}x)\|_4^4 \\ &\le\; C(t-s)(\|T_{s_{j+1}}x\|_N^3 \|AT_{s_{j+1}}x\|_2 + \|T_{s_{j+1}}x\|_N^2 \|\Gamma(T_{s_{j+1}}x, T_{s_{j+1}}x)\|) \; . \end{aligned}$$

This allows us to define the weak* limit

$$m_{s_0} \;=\; \lim_\sigma m_\sigma$$

as a martingale in $L_4(N)$ with continuous path such that

$$\begin{aligned} &\lim_\sigma \|m_\sigma\|^2_{H_p^c(\sigma)} \;\le\; c_{p/2} \lim_\sigma \|\sum_j \int_{s_j}^{s_{j+1}} \pi_r(\Gamma(T_rT_{s_{j+1}}x, T_rT_{s_{j+1}}x))\|_{p/2} \\ &\le c_{p/2} \lim_\sigma \|\sum_j \int_{s_j}^{s_{j+1}} \pi_r(T_r\Gamma(T_{s_{j+1}}x, T_{s_{j+1}}x))\|_{p/2} \\ &=\; c_{p/2} \lim_\sigma \|\sum_j \int_{s_j}^{s_{j+1}} E_{[r}\pi_0(\Gamma(T_{s_{j+1}}x, T_{s_{j+1}}x))\|_{p/2} \\ &\le c_{p/2}^2 \lim_\sigma \|\sum_j \int_{s_j}^{s_{j+1}} \Gamma(T_{s_{j+1}}x, T_{s_{j+1}}x)dr\|_{p/2} \;=\; c_{p/2}^2 \|\int_0^{s_0} \Gamma(T_sx, T_sx)ds\|_{p/2} \; . \end{aligned}$$

With the results from [JK], this implies that $y_{s_0} \in H_p^c$ for all $p < \infty$. Using the same estimate for x^*, we deduce that $m_{s_0} \in L_p$ for all $p < \infty$. We may take another weak* limit to define $m^2(x) = \lim_{s_0\to\infty} m_{s_0}$ which has continuous path because of the uniform estimate for the norm $\|x\|_{h_4^d}$. Moreover, the proof of

$$\langle m^2(x), m^2(x)\rangle \;=\; 2\int_0^\infty \pi_s(\Gamma(T_{2s}x, T_{2s}x))ds \; . \tag{2.5}$$

is the same as in Lemma 2.1. Even without knowing exactly what the bracket looks like, it is easy to complete the estimate for selfadjoint x. We assume that x has mean 0, i.e. $\Pr(x) = x$. Let $y \in \mathcal{A}$ such that $\|y\|_{p'} = 1$ and

$$\|x\|_p \;\le\; (1+\delta)|\tau(y^*x)| \; .$$

Let us consider

$$\begin{aligned} \tau(\pi_0(y)^*m_\sigma) \;&=\; \sum_j \tau\big((\pi_{s_j}(T_{s_j}(y^*)) - \pi_{s_{j+1}}(T_{s_{j+1}}(y^*)))d_j\big) \\ &= 2\sum_j \int_{s_j}^{s_{j+1}} \tau\Big(\pi_r\big(\Gamma(T_ry, T_{s_{j+1}+r}x)\big)\Big)dr \;=\; 2\sum_j \int_{s_j}^{s_{j+1}} \tau\Big(\Gamma(T_ry, T_{s_{j+1}+r}x)\Big)dr \; . \end{aligned}$$

Thus in the limit we obtain

$$\lim_{s_0}\lim_\sigma \tau(\pi_0(y)^*m_\sigma) \;=\; 2\int_0^\infty \tau(\Gamma(T_ry, T_{2r}x))dr \;=\; 2\int_0^\infty \tau(y^*AT_{3r}x)dr \; .$$

Using the spectral resolution for $A=\int_0^\infty \lambda dE(\lambda)$ and $d\nu_x(\lambda)=(x,dE(\lambda)x)$, we see that

$$\int_0^\infty (x,AT_{3r}x)dr=\int_0^\infty\int_0^\infty \lambda e^{-3r\lambda}drd\nu_x(\lambda) \;=\; \frac{1}{3}(E_{>0}x,E_{>0}x)\;.$$

Thus, $\int_0^\infty AT_{3r}x=\frac{1}{3}E_{>0}x$, where $E_{>0}=I-\mathrm{Pr}$ is the orthogonal projection onto the complement of of the kernel of A. Since for selfadjoint x the martingale $m^2(x)$ is also selfadjoint, we deduce

$$\begin{aligned}
\|x\|_p \;&\le\; 3(1+\delta)\lim_{s_0}\lim_{\sigma}|\tau(\pi_0(y^*)m_\sigma)| \;=\; 3(1+\delta)|\tau(\pi_0(y)^*m^2(x))|\\
&\le c_p3(1+\delta)\|y\|_{p'}\|m_2(x)\|_{H_p^c}\;\le\; c_pc_{p/2}3(1+\delta)\lim_{s_0}\lim_{\sigma}\|\sum_j\int_{s_j}^{s_{j+1}}\Gamma(T_{s_{j+1}}x,T_{s_{j+1}}x)dr\|_{p/2}^{1/2}\\
&= c_pc_{p/2}3(1+\delta)\;\|(\int_0^\infty \Gamma(T_sx,T_sx)ds)^{\frac12}\|_p\;.
\end{aligned}$$

For the upper estimate we refer to [Jun08] which applies due to H^∞-calculus. ■

Lemma 2.4. *Let (T_t) and $\mathcal{A}$ be as above. Assume that there is a further von Neumann algebra M, a sequence $(u_j):\mathcal{A}\to\mathcal{M}$ such that*

$$\Gamma(x,x)\;=\;\sum_j u_j(x)^*u_j(x)$$

and semigroup $\hat{T}_t$ with cb-H^∞ calculus such that

$$(u_j(T_tx))\;=\;(\hat{T}_tu_j(x))\;.$$

Then

$$\|\int_0^\infty \Gamma(A^{1/2}T_tx,A^{1/2}T_tx)dt\|_{p/2}\;\le\; c_p^2\;\|\Gamma(x,x)\|_{p/2}$$

holds for all $2\le p<\infty$ and all mean 0 elements $x\in\mathcal{A}$.

Proof. Since $id_{B(\ell_2)}\otimes\hat{T}_t$ satisfies H^∞-calculus, we deduce from [JLMX06] that

$$\begin{aligned}
(2.6)\qquad \|\int_0^\infty \Gamma(A^{1/2}T_tx,A^{1/2}T_tx)dt\|_{p/2}^{1/2}\;&=\;\|(\int_0^\infty |\hat{A}^{1/2}\hat{T}_t(\sum_j e_{j,1}\otimes u_j(x))|^2dt)^{1/2}\|_p\\
&\le c_p\|\sum_j e_{j,1}\otimes u_j(x)\|_{L_p(B(\ell_2)\otimes M)}\;=\;c_p\|\Gamma(x,x)^{\frac12}\|_p\;.
\end{aligned}$$

Here $\hat{A}$ is the generator of $\hat{T}_t$. Note also that $u_j(T_tx)=\hat{T}_tu_j(x)$ implies $u_j(P_tx)\;=\;\hat{P}_tu_j(x)$ according to (1.8) and hence by differentiation $\hat{A}^{1/2}u_j(T_tx)=u_j(A^{1/2}T_tx)$. This justifies the first equation in (2.6) and completes the proof. ■

Let us recall the notation $\Gamma_{\alpha_1,\alpha_2}=(\Gamma_{A^{\alpha_1}})_{A^{\alpha_2}}$ for iterated gradients, see [Jun08].

Proposition 2.5. *Let (T_t) be a semigroup with a Markov dilation and $\Gamma^2\;\ge\;0$. Let (P_t) be the associated Poisson semigroup satisfying*

$$A^{1/2}\mathcal{A}\subset\mathcal{A}\quad,\quad P_s(\mathcal{A})\subset\mathcal{A}$$

and that $f(s,t)=P_s\Gamma_{1/2,1}(P_tx,P_tx)$ is L_2-continuous for $x\in\mathcal{A}$. Then

$$\|(\int_0^\infty \Gamma(A^{1/2}P_sx,A^{1/2}P_sx)sds)^{\frac12}\|_p\;\le\; c_p\|\Gamma(x,x)^{\frac12}\|_p\;.$$

Proof. A glance at (1.8) shows a Markov dilation for T_t implies that P_t is factorable. According to [CL99], we deduce that P_t also has a Markov dilation. Let us denote this family of maps again with π_s. We consider the submartingale

$$y_s \;=\; \pi_s(\Gamma(P_sx,P_sx))\;.$$

Indeed, for $s<t$ we deduce from $\Gamma^2 \;\ge\; 0$ that

$$E_{[t}(y_s) \;=\; \pi_t(P_{t-s}\Gamma(P_sx,P_sx)) \;\ge\; \pi_t(\Gamma(P_tx,P_tx))\;.$$

As in Lemma 2.1 we consider $f(r)=P_{t-r}\Gamma(P_rx,P_rx)$ and obtain

$$\begin{aligned} f'(r) &= A^{1/2}P_{t-r}\Gamma(P_rx,P_rx)-P_{t-r}\Gamma(A^{1/2}P_rx,P_rx)-P_{t-r}\Gamma(P_rx,A^{1/2}P_rx)\\ &= -2P_{t-r}\Gamma_{1/2,1}(P_rx,P_rx)\;. \end{aligned}$$

This implies

$$E_{[t}(y_s-y_t) \;=\; 2E_{[t}\int_s^t \pi_r(\Gamma_{1/2,1}(P_rx,P_rx))dr\;.$$

We apply (2.4) for P_s and obtain

$$\|\pi_r(y)-\pi_t(P_{t-r}y)\|_2^2 \;\le\; \|y\|_2\|P_r(y)-P_t(y)\|_2\;.$$

We deduce that

$$\begin{aligned} &\|\sum_j\int_{s_j}^{s_{j+1}} \pi_r(\Gamma_{1/2,1}(P_rx,P_rx))-E_{s_{j+1}}(\pi_r(\Gamma_{1/2,1}(P_rx,P_rx))dr\|\\ &\le \sum_j\int_{s_{j+1}}^{s_j} \|\Gamma_{1/2,1}(P_rx,P_rx)\|_2^{1/2}\|(P_r-P_{s_{j+1}})\Gamma_{1/2,1}(P_rx,P_rx)\|_2^{1/2}dr\;. \end{aligned}$$

Thus uniform continuity of $P_t\Gamma_{1/2,1}(P_rx,P_rx)$ implies that the limit converges to 0 as long as the mesh size converges to 0. Therefore we obtain

$$\lim_{|\sigma|\to 0}\sum_j E_{s_{j+1}}(y_{s_j}-y_{s_{j+1}}) \;=\; 2\int_s^t \pi_r(\Gamma_{1/2,1}(P_rx,P_rx))dr\;.$$

Now, we apply the inequality for potentials Lemma 1.2 and find

$$\|\sum_j E_{s_{j+1}}(y_{s_j}-y_{s_{j+1}})\|_{p/2} \;\le\; c_{p/2}\|y_0\|_{p/2}\;.$$

Indeed, since we are working with a reversed martingale, $y_0=\pi_0(\Gamma(x,x))$ is the endpoint. Passing to the limit, we deduce that

$$\|\int_0^\infty \pi_s(\Gamma_{1/2,1}(P_sx,P_sx))ds\|_{p/2} \;\le\; c_{p/2}\|y_0\|_{p/2} \;=\; c_p\|\Gamma(x,x)\|_{p/2}\;.$$

Conditioning on π_0 yields

$$\|\int_0^\infty P_s\Gamma_{1/2,1}(P_sx,P_sx)ds\|_{p/2} \;\le\; c_{p/2}\|\Gamma(x,x)\|_{p/2}\;.$$

Now, we recall from [Jun08] that

$$\begin{aligned} \Gamma_{1/2,1}(y,y) \;&=\; \int_0^\infty P_t\Gamma(A^{1/2}P_ty,A^{1/2}P_ty)dt+\int_0^\infty \Gamma^2(P_ty,P_ty)dt\\ &\ge\; \int_0^\infty P_t\Gamma(A^{1/2}P_ty,A^{1/2}P_ty)dt\;. \end{aligned}$$

Here we use $\Gamma^2 \geq 0$. Therefore, we obtain

$$\begin{aligned}
&\| \int_0^\infty \Gamma(A^{1/2}P_s x, A^{1/2}P_s x) s ds\|_{p/2} \;=\; 2\,\| \int_0^\infty \Gamma(A^{1/2}P_{2s}x, A^{1/2}P_{2s}x) 2s ds\|_{p/2} \\
&\leq 2\,\| \int_0^\infty P_s\Gamma(A^{1/2}P_s x, A^{1/2}P_s x)2s ds\|_{p/2} \\
&= 2\,\| \int_0^\infty \int_0^\infty P_{s+t}\Gamma(A^{1/2}P_{s+t}x, A^{1/2}P_{s+t}x) ds dt\|_{p/2} \\
&\leq\; 2\,\| \int_0^\infty P_s\Gamma_{1/2,1}(P_s x, P_s x)) ds\|_{p/2} \;\leq\; 2c_{p/2}\, \|\Gamma(x,x)\|_{p/2}\, . \qquad \blacksquare
\end{aligned}$$

Our next task is to replace P_s by T_s following a well-known path in [JLMX06].

Lemma 2.6. *Let $2 \leq p < \infty$, $\tilde{\Gamma}$ be a completely positive form on $\bar{\mathcal{A}} \times \mathcal{A}$, and (T_t) be a semigroup of selfadjoint maps with selfadjoint generator such that*

$$\|(\sum_k \tilde{\Gamma}(T_{z_k}x_k, T_{z_k}x_k))^{1/2}\|_p \;\leq\; c(p,\theta)\|(\sum_k \tilde{\Gamma}(x_k, x_k))^{1/2}\|_p \tag{2.7}$$

for all z_k with $0 \leq Arg(z_k) \leq \theta$, where $0 < \theta < \pi$. Moreover, assume that $A^{1/2}L_2(N)$ is dense in $(I - \mathrm{Pr})L_2(N)$ with respect to $\|x\|_{\tilde{\Gamma}} = \tau(\tilde{\Gamma}(x,x))$. Then

$$\|(\int_0^\infty \tilde{\Gamma}(T_s x, T_s x) ds)^{\frac{1}{2}}\|_p \;\leq\; c_0 c(p,\theta)\|(\int_0^\infty \tilde{\Gamma}(P_s x, P_s x) s ds)^{\frac{1}{2}}\|_p \tag{2.8}$$

holds for all $x \in \mathcal{A}$ with $\mathrm{Pr}(x) = 0$.

Proof. We introduce the space $L_p(L_2^c(\tilde{\Gamma}))$ as the the closure of continuous functions such that

$$\|f\|_{L_p(L_2^c(\tilde{\Gamma}))} \;=\; \|(\int_0^\infty \tilde{\Gamma}(f(s), f(s))\, \frac{ds}{s})^{1/2}\|_p\, .$$

As in [JLMX06, Corollary 4.9] our assumption implies that the family $z(z-a)^{-1}$ is Col-bounded on $L_p(L_2^c(\Gamma))$ for the same angle. Then we may apply [JLMX06, Theorem 4.14] and deduce that T_Φ with the kernel $\Phi(s,t) = F_2(sA)F_1(tA)$ is bounded on $L_p(L_2(\tilde{\Gamma}))$. We may choose $F_2(z) = z^{1/2}e^{-z}$ and $F_1(z) = ze^{-z}$. Let us assume that $x = A^{1/2}y$. Let us define $f(t) = \sqrt{At}P_{\sqrt{t}}y$ Using a change of variable, we deduce that

$$\int_0^\infty \tilde{\Gamma}(f(t), f(t))\, \frac{dt}{t} = \int_0^\infty \tilde{\Gamma}(P_{\sqrt{t}}x, P_{\sqrt{t}}x) t\, \frac{dt}{t} \;=\; \frac{1}{2}\int_0^\infty \tilde{\Gamma}(P_s x, P_s x)\, s^2 \frac{ds}{s}\, .$$

In order to apply T_Φ we have to calculate

$$\int_0^\infty F_1(tA)f(t)\, \frac{dt}{t} = \int_0^\infty tAT_t(t^{1/2}A^{1/2}P_{\sqrt{t}}y)\, \frac{dt}{t} \;=\; \int_0^\infty t^{\frac{3}{2}}A^{\frac{3}{2}}T_t P_{\sqrt{t}}y\, \frac{dt}{t}\, .$$

Let dE_λ be the spectral measure for A and $d\mu_{y_1,y}$ the induced probability measure for elements $y_1, y \in L_2(N)$. Then

$$\begin{aligned}
\tau(y_1^* \int_0^\infty t^{\frac{3}{2}}A^{\frac{3}{2}}T_t P_{\sqrt{t}}y\, \frac{dt}{t}) &= \int_0^\infty \int_0^\infty t^{\frac{3}{2}}\lambda^{\frac{3}{2}}e^{-t\lambda}e^{-\sqrt{t}\sqrt{\lambda}}\frac{dt}{t}d\mu_{y_1,y}(\lambda) \\
&= \left(\int_0^\infty t^{\frac{3}{2}}e^{-t}e^{-\sqrt{t}}\frac{dt}{t}\right)\tau(y_1^* y_2)\, .
\end{aligned}$$

Let us denote by c the constant given by the converging integral. Since y_1 is arbitrary we deduce by approximation that

$$T_\Phi(f)(s) \;=\; cA^{1/2}T_s(y) \;=\; cT_s(x)$$

holds for all $x \in L_2(L_2(\tilde{\Gamma}))$ with $\Pr(x) = 0$. Let us note that the assumption that A is selfadjoint is not necessary. In the sectorial case we refer to [JLMX06, Lemma 6.5] and the argument in [JLMX06, Theorem 6.7]. The assertion follows from the boundedness of T_Φ on $L_p(L_2(\tilde{\Gamma}))$. ■

Remark 2.7. For a semigroup of (selfadjoint) completely positive maps and the canonical form $\tilde{\Gamma}(x,y) = x^*y$, we deduce that

$$\|x\|_{H_p^c(T)} \;\le\; c(p)\|x\|_{H_p^c(P)}$$

without assuming H^∞-calculus. The reverse inequality is shown in [Jun08] and hence the equivalence of different semigroup H_p-norms holds without using a Markov dilation.

Remark 2.8. Assume the assumption of Lemma 2.4 is satisfied. Then (2.7) holds for $\tilde{\Gamma} = \Gamma$, $\theta < \pi(\frac{1}{2} - \frac{1}{p})$ and (2.8) holds also.

Proof. For a selfadjoint semigroup the results in [JX03] imply

$$\|\sum_k |\hat{T}_{t_k} y_k|^2\|_{p/2} \;\le\; \|\sum_k \hat{T}_{t_k} |y_k|^2\|_{p/2} \;\le\; c_p\|\sum_k |y_k|^2\|_{p/2} \,.$$

For $p = 2$ we have

$$\|\sum_k |\hat{T}_{z_k} y_k|^2\|_1 \;=\; \sum_k \|\hat{T}_{z_k} y_k\|_2^2 \;\le\; \sum_k \|y_k\|_2^2$$

whenever $Re(z_k) \;\ge\; 0$. Then the assertion is a standard application of Stein's theorem on interpolation of analytic families applied to $y_k = u(x_k)$ and yields (2.7). Moreover, we may directly apply the argument in Lemma 2.6 for $\hat{T}_t$ and (y_j). Note that x orthogonal to the kernel of A implies that (y_j) is orthogonal to the kernel of $\hat{A}$. Thus we obtain (2.8) without using the extra density assumption. ■

The following argument is based on a continuous version of a result of Stein.

Theorem 2.9. *Let $1 < p < \infty$.*

i) *Let $m = \int_0^\infty dm_s$ be a martingale with continuous path in H_p. Then*

$$\|\left(\int_0^\infty |\frac{1}{r}\int_0^r s\, dm_s|^2 \frac{dr}{r}\right)^{1/2}\|_p \;\le\; c_p\|m\|_{H_p^c} \,.$$

ii) *Let (T_t) be semigroup with a martingale dilation and $x \in \mathcal{A}$. Then*

$$\|x\|_{H_p^c(T)} \;\le\; c_p\|\pi_0(x)\|_{H_p^c} \,.$$

iii) *Let in addition $2 \le p < \infty$ and $\Gamma^2 \;\ge\; 0$. Then*

$$c_p^{-1}\|x\|_{H_p^c(T)} \;\le\; \|\left(\int_0^\infty \Gamma(T_s x, T_s x) ds\right)^{\frac{1}{2}}\|_p \;\le\; C_p\|x\|_{H_p^c(T)} \,.$$

Proof. The bulk of the argument is due to Stein, the noncommutative part of the argument is contained in [JLMX06, Proposition 10.8] where it is proved that for a martingale difference sequence (d_j),

$$E_k \;=\; \sum_{j=0}^k d_j \quad , \quad \Lambda_m(x) \;=\; \frac{1}{m+1}\sum_{k=0}^m E_k(x)$$

and $\Delta_m(x) = \Lambda_m(x) - \Lambda_{m-1}(x)$ one has

$$\|(\sqrt{m}\Delta_m(x))\|_{L_p(\ell_2^c)} \;\le\; C_p\|(d_j)_j\|_{L_p(\ell_2^c)} + C_p\|(\sum_{j=2^k+1}^{2^{k+1}} d_j)_k\|_{L_p(\ell_2^c)} \,. \tag{2.9}$$

Here C_p is the norm of Stein's projection in L_p. It is also important to note that this argument is true for decreasing or increasing martingale differences. Moreover, let (m_t) be a continuous martingale so that

$$\lim_{|\sigma|\to 0} \|(\sum_{t_j} |m_{t_{j+1}} - m_{t_j}|^2)^{1/2}\|_p \;=\; \|m\|_{H_p^c} \,.$$

Here the limit is taken over partitions of a fixed interval $[\alpha,\beta]$ and the mesh size of the partitions converges to 0. Then we note that the right hand side of (2.9) is controlled by two partitions and hence

$$\lim_{|\pi|\to 0} \|(\sqrt{l}\Delta_l(m))\|_{L_p(\ell_2^c)} \;\le\; C_p\|m\|_{H_p^c} \,.$$

Let us now fix $0<\alpha<\beta<\infty$ and assume that

$$E_k(x) \;=\; \int_\alpha^{\alpha+\frac{k}{n}} dx_s$$

holds in terms of stochastic integrals. This implies

$$\begin{aligned}
\Lambda_l(m) - \Lambda_{l-1}(m) \;&=\; \frac{1}{l+1}E_l(m) + \sum_{k=1}^{l-1} E_k(m)(\frac{1}{l+1} - \frac{1}{l}) \\
&= \frac{1}{l+1}\int_{\alpha+\frac{l-1}{n}}^{\alpha+\frac{l}{n}} dm_s + \frac{1}{l+1}\int_\alpha^{\alpha+\frac{l-1}{n}} (1-(\frac{l-\lceil n(s-\alpha)\rceil}{l}))dm_s \\
&= \frac{1}{l+1}\int_{\alpha+\frac{l-1}{n}}^{\alpha+\frac{l}{n}} dm_s + \frac{1}{l(l+1)}\int_\alpha^{\alpha+\frac{l-1}{n}} \lceil n(s-\alpha)\rceil dm_s \,.
\end{aligned}$$

Here $\lceil x\rceil$ is the smallest integer $\ge x$. The first part is easy to control and hence

$$\|\sum_{l=2}^{\frac{\alpha-\beta}{n}} \frac{1}{l-1}|\int_\alpha^{\alpha+\frac{l-1}{n}} \frac{\lceil n(s-\alpha)\rceil}{l-1} dm_s|^2\|_{p/2}^{1/2} \;\le\; c_p\|x\|_{H_p^c} \,.$$

Passing to the limit (see [JLMX06,] for a similar reasoning), we deduce that

$$\|\left(\int_\alpha^\beta |\int_\alpha^r \frac{(s-\alpha)}{r} dm_s|^2 \frac{dr}{r}\right)^{\frac12} \|_p \;\le\; c_p\|x\|_{H_p^c}$$

provided the square function is Riemann integrable. Finally, we may sent α and β to the boundary and obtain

$$\|\left(\int_0^\infty |\frac{1}{r}\int_0^r s dm_s|^2 \frac{dr}{r}\right)^{\frac12} \|_p \;\le\; c_p\|m\|_{H_p^c} \,.$$

This completes the proof of i). We apply this inequality first in the particular case where $m=\pi_0(x)=\int_0^\infty dm_s(x)$ is the reversed martingale decomposition. We also add the conditional expectation E_0. Let $\pi_0(y)=\int_0^\infty dm_s(y)$ an other element. Then we deduce from the calculus of brackets that

$$\begin{aligned}
\tau(y^*\int_0^r \frac{s}{r} dm_s(x)) \;&=\; 2\int_0^r \tau(\pi_s(\Gamma(T_s y, T_s x)\frac{s}{r}))ds \\
&= 2\int_0^r \tau(y^* A T_{2s} x)\frac{s}{r}))ds \;=\; 2\tau(y^*\int_0^r AT_{2s}x)\frac{s}{r}ds \;=\; \tau(y^*\int_0^1 AT_{2sr}x2sr\; ds) \,.
\end{aligned}$$

Let us define

$$f(z) \;=\; \int_0^1 (2zs)e^{-2sz}ds \;=\; \frac{1-e^{-2z}}{2z} - e^{-2z} \,.$$

Note that $f(0) = 0$ and vanishes at ∞. Then we see that

$$f(rA)x \;=\; \int_0^1 2rsAT_{2rs}xds \;.$$

Using the equivalence of different square functions [JLMX06], we deduce that

$$\|x\|_{H_p^c(T)} \;\le\; c_p \,\|\pi_0(x)\|_{H_p^c} \;.$$

For our last assertion we consider $p \;\ge\; 2$ and the martingale $m = m^2(x)$. For fixed r, we deduce from the fact that the function $f(s) = s$ is adapted that

$$\int_0^t s\; dm_s^2(x)$$

is a martingale and hence

$$\langle \int_0^\infty dm_s(y), \int_0^r dm_s^2(x)\, s\rangle \;=\; 2\int_0^r \pi_s(\Gamma(T_sy, T_{2s}x))sds \;.$$

Here we use the projection on $\pi_0(N)$ from Lemma 2.3 and the calculus of brackets for stochastic integrals. This implies

$$E_0(\frac{1}{r}\int_0^r sdm_s^2(x)) \;=\; \frac{2}{r}\int_0^r AT_{3s}x\; sds \;=\; \frac{2}{3}\int_0^1 AT_{3sr}x\;(3sr)ds \;.$$

Thus replacing f by $\tilde{f}(z) = \int_0^1 (3sz)e^{-3sz}ds$, we deduce again with the equivalence of different square functions that

$$\|x\|_{H_p^c(T)} \;\le\; c_p\|m^2(x)\|_{H_p^c} \;\le\; c_p^2\,\|\left(\int_0^\infty \Gamma(T_sx, T_sx)ds\right)^{\frac{1}{2}}\|_p \;.$$

The last estimate is of course taken from Proposition 2.3. Assuming $\Gamma^2 \;\ge\; 0$, we can refer to [Jun08] for the estimate

$$\|\left(\int_0^\infty \Gamma(T_sx, T_sx)ds\right)^{\frac{1}{2}}\|_p \le c_p\|x\|_{H_p^c(T)} \;. \qquad \blacksquare$$

Theorem 2.10. *Let (T_t) be a semigroup satisfying $\Gamma^2 \;\ge\; 0$. Assume that*

i) *The assumption of Lemma 2.4 is satisfied* or
ii) *Condition (2.8) is satisfied for $\widetilde{\Gamma} = \Gamma$ and $f(t,s) = P_s\Gamma_{1/2,1}(P_tx, P_tx)$ is L_2-continuous for $x \in \mathcal{A}$.*

Then

$$\|A^{\frac{1}{2}}x\|_{H_p^c(T)} \;\le\; c_p\,\|\Gamma(x,x)^{\frac{1}{2}}\|_p$$

and

$$\|A^{\frac{1}{2}}x\|_p \;\le\; c_p \max\{\|\Gamma(x,x)^{\frac{1}{2}}\|_p, \|\Gamma(x^*,x^*)^{\frac{1}{2}}\|_p\} \;.$$

holds for all mean 0 elements $x \in \mathcal{A}$.

Proof. For the first assertion we combine Theorem 2.9iii) with Lemma 2.4 or Remark 2.8. This allows us to apply Proposition (2.5). For the second assertion, we refer to [JLMX06] for the fact that a Markov dilation implies H^∞-calculus and hence

$$\|A^{1/2}x\|_p \sim_{c_p} \|A^{1/2}x\|_{H_p(T)} \;=\; \max\{\|A^{1/2}x\|_{H_p^c(T)}, \|A^{1/2}x^*\|_{H_p^c(T)}\} \;.$$

This immediately implies the second assertion. $\blacksquare$

Remark 2.11. In a forthcoming publication we will show that the assumption in Lemma 2.4 is satisfied for Fourier multipliers on discrete groups. Motivated by the recent work of Ricard we will also construct a Markov dilation satisfying the conditions i) and ii) at the beginning of this section.

3. The probabilistic model

The probabilistic approach to Littlewood-Paley theory goes back to the work of P.A. Meyer and has found many applications. Instead of adding a time component to the manifold as in Stein's approach, the probabilistic approach adds an additional brownian motion to the picture. As in the previous section we assume that (T_t) is a semigroup of completely positive maps and that π_s is a Markov dilation satisfying i). (The reversed condition is no longer necessary). Let us keep the notation Pr_0 for the projection on the kernel of A and start with a simple observation, well-known in the commutative case.

Lemma 3.1. *Let $-A$ be the negative generator of $T_t = e^{-tA}$ and $x \in \mathrm{dom}(A)$ with $\mathrm{Pr}_0(x) = 0$. Then*

$$m_t(x) \;=\; \pi_t(x) + \int_0^t \pi_s(Ax)ds$$

is a martingale.

Proof. For $p > 0$ we calculate

$$E_s(\int_0^\infty e^{-pt}\pi_t((p+A)x)dt) \;=\; \int_0^s e^{-pt}\pi_t((p+A)x)dt + \int_s^\infty (p+A)e^{-pt}\pi_s(T_{t-s}(p+A)x)dt$$
$$= \int_0^s e^{-pt}\pi_t((p+A)x)dt + e^{-ps}\pi_s(\int_0^\infty e^{-t(p+A)}(p+A)xdt) \;.$$

A change of variables shows that $\int_0^\infty e^{-t(p+\lambda)}(p+\lambda)dt = 1$ holds for every $\lambda \in \mathbb{R}$. Thus (arguing in L_2 if necessary), we see that $\int_0^\infty e^{-t(p+A)}(p+A)xdt = x$ for all x with $\mathrm{Pr}_0(x) = 0$. Hence

$$m_{s,p} \;=\; e^{-ps}\pi_s(x) + \int_0^s e^{-pt}\pi_t(px + Ax)dt$$

is a martingale for all $p > 0$. Sending $p \to 0$ implies that $m_s(x)$ is a martingale. ■

Lemma 3.2. *Let $-A$ be the generator of $T_t = e^{-tA}$ and Γ the associated gradient form. Assume that the filtration M_s is continuous. Let $x, y \in \mathcal{A}$. Then*

$$\langle m(x), m(y)\rangle_t \;=\; 2\int_0^t \pi_s(\Gamma(x,x))ds \;.$$

Proof. Let us recall that for adapted process (a_s) and (b_s) we have

$$\langle a, b\rangle_s \;=\; \lim_{\pi,\mathcal{U}} \sum_j E_{s_j}((a_{s_{j+1}} - a_{s_j})^*(b_{s_{j+1}} - b_{s_j})) \;.$$

The limit is taken in the weak sense. In particular, the bracket is bilinear and vanishes on martingales, because then $E_{s_j}(m_{s_{j+1}} - m_{s_j}) = 0$. It is best to start with

$$\pi_t(x^*x) - \pi_s(x^*x) = m_t(x^*x) - m_s(x^*x) - \int_s^t \pi_r(A(x^*x))dr \;.$$

We use the notation $\pi_t(x) = m_t(x) + a_t(x)$ where $a_t(x) = \int_0^t \pi_r(-Ax)dr$ and $m_t(x)$ is the martingale from the previous Lemma 3.1. Then we find

$$\pi_t(x^*x) - \pi_s(x^*x) \;=\; \pi_t(x)^*\pi_t(x) - \pi_s(x)^*\pi_s(x)$$

$$
\begin{aligned}
&= (\pi_t(x)-\pi_s(x)+\pi_s(x))^*(\pi_t(x)-\pi_s(x)+\pi_s(x))\\
&= (\pi_t(x)-\pi_s(x))^*\pi_s(x)+\pi_s(x)^*(\pi_t(x)-\pi_s(x))+(\pi_t(x)-\pi_s(x))^*(\pi_t(x)-\pi_s(x))\\
&= (m_t(x)-m_s(x))^*\pi_s(x)+(a_t(x)-a_s(x))\pi_s(x)\\
&\quad +\pi_s(x)^*(m_t(x)-m_s(x))+\pi_s(x)^*(a_t(x)-a_s(x))\\
&\quad +(m_t(x)-m_s(x))^*(m_t(x)-m_s(x))+(m_t(x)-m_s(x))^*(a_t(x)-a_s(x))\\
&\quad +(a_t(x)-a_s(x))(m_t(x)-m_s(x))+(a_t(x)-a_s(x))^*(a_t(x)-a_s(x))\ .
\end{aligned}
$$

After applying E_s the first and the third term disappear. Then we observe that

$$\|(a_t(x)-a_s(x))^*(a_t(x)-a_s(x))\| \ \le\ (t-s)^2\sup_r\|\pi_r(Ax)\|$$

and hence for bounded Ax this terms disappears when the mesh size goes to 0. Since we assume that the filtration is continuous, we know that $m_s(x)$ is norm continuous in L_{2p}, $p<\infty$. Thus by uniform continuity we find

$$\lim_{\delta\to 0}\sup_{|s_{j+1}-s_j|<\delta}\sum_j\|(a_{s_{j+1}}(x^*)-a_{s_j}(x^*))(m_{s_{j+1}}(x)-m_{s_j}(x))\|_p \ =\ 0\ .$$

Note that the L_p continuity of m_s implies the L_p continuity of π_s. Therefore we obtain in the limit (in the Riemann sense)

$$\langle m(x),m(x)\rangle_t \ =\ -\int_0^t\pi_r(A(x^*x))dr+\int_0^t\pi_r(Ax^*)\pi_r(x)dr-\int_0^t\pi_r(x^*)\pi_r(Ax)dr\ .$$

Here we use (2.4) for $a=\Gamma(x,x)$. By assumption $A\Gamma(x,x)\in\mathcal{A}$ for $x\in\mathcal{A}$ and hence we $\pi_r(a)-E_{s_j}\pi_r(a)$ goes do 0 uniformly in $|r-s_{j+1}|$. By polarization the formula is true for all x,y. ■

The main ingredient in the probabilistic approach towards Riesz transforms is to use Lévy's stopping time argument for the Brownian motion (see however [Gun86], [GV79] for more compact notation). Let (B_t) be a classical brownian motion with generator ds (instead of the usual $\frac{1}{2}ds$) such that $B_0=a$ holds with probability 1. Then we consider the stopping time $\mathbf{t}_a=\inf\{t: B_t(\omega)=0\}$. Instead of $\mathcal{A}$ we consider now the tensor product $\mathcal{A}(B)\otimes\mathcal{A}$ where $\mathcal{A}(B)$ is the algebra of polynomials in the variables B_t. The new generator is

$$\hat{A}\ =\ -\frac{d^2}{dt^2}+A\ .$$

This leads to

$$\hat{\Gamma}(x,y)\ =\ \Gamma(x,y)+\frac{d}{dt}x^*\frac{d}{dt}y\ .$$

The Markov dilation is given by $\hat{\pi}_t(f\otimes x)\ =\ f(B_t)\otimes\pi_t(x)$. Indeed, let $\hat{M}_s\ =\ M_s^B\otimes M_s$ be the von Neumann algebra given by the tensor product of the Brownian motion observed until time s and the von Neumann algebra M_s given by the Markov dilation. Then

$$
\begin{aligned}
\hat{E}_s(f(B_t)\otimes\pi_t(x)) = E_s^B(f(B_t))E_s(\pi_t(x)) \ &=\ T^B_{t-s}f(B_s)\,\pi_s(T_{t-s}(x))\\
&= \hat{\pi}_s((T^B_{t-s}\otimes T_{t-s})(f\otimes x))\ .
\end{aligned}
$$

Here we use the fact that the brownian motion is the Markov process for the generator $D^2=\frac{d^2}{dt^2}$ with corresponding semigroup $T^B_t=e^{tD^2}$. For an element $x\in\mathcal{A}$ we use the notation $Px\in L_\infty(\mathbb{R}_+;N)$ given by the function

$$Px(t)\ =\ P_t(x)\ .$$

We will also write $P'x$ for the function $\frac{d}{dt}P_tx$. Harmonicity now leads to a well-known martingale property (again due to Meyer).

Proposition 3.3. *Let $\mathbf{t}_a$ be the stopping time as above. Then*

$$n_t(x) \ = \ \hat{\pi}_{\mathbf{t}_a \wedge t}(Px)$$

is a martingale with bracket

$$\langle n(x), n(x)\rangle_t \ = \ 2\int_0^{\mathbf{t}_a\wedge t} \hat{\pi}_s(\hat{\Gamma}(Px,Px))ds \ .$$

Proof. We consider $y = \hat{\pi}_{\mathbf{t}_a}(Px) \ = \ \pi_{\mathbf{t}_a}(x)$. Let us calculate the conditional expectation $\hat{E}_s$:

$$\hat{E}_s(\pi_{\mathbf{t}_a}(x)) = 1_{\mathbf{t}_a\le s}\hat{E}_s(\pi_{\mathbf{t}_a}(x)) + 1_{\mathbf{t}_a>s}\hat{E}_s(\pi_{\mathbf{t}_a}(x)) \ = \ 1_{\mathbf{t}_a\le s}\pi_{\mathbf{t}_a}(x) + 1_{\mathbf{t}_a>s}E_s^B(\pi_s(T_{\mathbf{t}_a-s}(x)))$$

Now, we fix an $\omega\in\Omega$ such that $B_s(\omega)=b$ and $s<\mathbf{t}_a$. This means $b>0$. Then $\mathbf{t}_a - s$ is exactly stopping time until $B_t - B_s$ hits 0. Let us recall that (see [IM74, page=25])

$$\mathbb{E}e^{-\lambda\mathbf{t}_b} \ = \ e^{-\sqrt{\lambda}b} \ .$$

Using the spectral resolution $A = \int \lambda dE(\lambda)$ we get

$$(x, \mathbb{E}T_{\mathbf{t}_b}(y)) \ = \ \mathbb{E}\int_0^\infty e^{-\lambda\mathbf{t}_b}d\nu_{x,y}(\lambda) \ = \ \int_0^\infty e^{-b\sqrt{\lambda}}d\nu_{x,y}(\lambda) \ = \ (x, P_b(x)) \ .$$

By continuity,

$$\mathbb{E}T_{\mathbf{t}_b}(y) \ = \ P_b y \tag{3.1}$$

holds L_p's. Hence we find

$$\mathbb{E}(\pi_s(T_{\mathbf{t}_a-s}(x))|B_s=b) \ = \ \pi_s(P_{B_s}(x)) \ = \ \hat{\pi}_s(Px) \ .$$

This proves the first assertion. For the second, we recall that

$$m_t(Px) \ = \ \hat{\pi}_t(Px) + \int_0^t \hat{\pi}_s(\hat{A}Px)ds$$

is a martingale and according to Lemma 3.2 we have

$$\langle m(x), m(x)\rangle_t \ = \ 2\int_0^t \hat{\pi}_s\hat{\Gamma}(Px,Px)ds \ .$$

Indeed, we may approximate Px by function of the form $\sum_j f_j\otimes x_j$ in the graph norm of $\hat{A}$ such that $f_j(s) \ = \ 0$ for $s<0$ and the apply Lemma 3.2. So that we read $P_s x = 1_{s>0}P_s x$. However, we have

$$\frac{d^2}{dt^2}(P_t(x)) \ = \ AP_t x \ .$$

and hence $\hat{A}Px = 0$. (This might no longer be true for the approximations but it holds in the limit). Thus $m = (\hat{\pi}_t(Px)_t)$ is martingale such that

$$\langle m, m\rangle_t \ = \ 2\int_0^t \hat{\pi}_s\hat{\Gamma}(Px,Px)ds \ .$$

Thus by conditioning on the stopping time $\mathbf{t}_a$ we still have

$$\langle n, n\rangle_{t\wedge\mathbf{t}_a} = 2\int_0^{t\wedge\mathbf{t}_a} \hat{\pi}_s\hat{\Gamma}(Px,Px)ds \ . \qquad \blacksquare$$

Remark 3.4. Using the stopping $\mathbf{t}_a$ we can explicitly construct a reversed Markov dilation for P_s once we have one for T_t. Indeed, let B be a brownian motion such that $\mathrm{Prob}(B_0 = \infty) = 1$ and $\mathbf{t}_b = \inf\{t : B_t(\omega) = b\}$. Such a random variable can be constructed using the limit for $a \to \infty$ of the finite brownian motions above. Then the random variable

$$\pi_b(x) \ = \ \pi_{\mathbf{t}_b}(x)$$

satisfies

$$E_c(\pi_b(x))(\omega) \ = \ \mathbb{E}(\pi_c(T_{\mathbf{t}_{c-b}}(x))|\mathbf{t}_c(\omega) = c)(\omega) \ = \ \pi_b(P_{c-b}x)(\omega)$$

for all $c > b$. Here we use the random filtration $M_{\mathbf{t}_c]}$.

In the following we fix the notation $\rho_a x = \pi_{\mathbf{t}_a} x$ for the induced trace preserving *-homomorphism. For $\kappa > 0$ we follow a similar idea as in section 1 and construct martingales $\rho_a^\kappa(x)$ such that

$$\langle \rho_a^\kappa(x), \rho_a^\kappa(x) \rangle_t \ = \ \int_0^{t\wedge \mathbf{t}_a} \hat{\pi}_s(\hat{\Gamma}(P^\kappa x, P^\kappa x))ds \ .$$

Here $P_s^\kappa(x) = P_{\kappa s}x$. Indeed, we fix a partition $\sigma = \{t_0, ..., t_n\}$ and define

$$m_\sigma \ = \ \sum_{j=1}^n \hat{E}_{t_{j+1}} \pi_{\rho_a}(P_{B_{t_j}}^{\kappa-1}(x)) - \hat{E}_{t_j} \pi_{\rho_a}(P_{B_{t_j}}^{\kappa-1}(x)) \ .$$

According to Lemma 3.3 we obtain

$$\langle m_\sigma, m_\sigma \rangle_t \ = \ 2 \sum_{t_{j+1}\le t} \int_{t_j}^{t_{j+1}} \pi_s \hat{\Gamma}(P^\kappa(B_{t_j}x), P^\kappa(B_{t_j}x))ds + \int_{t_{j_t}}^t \pi_s \hat{\Gamma}(P^\kappa(B_{t_{j_t}}x), P^\kappa(B_{t_{j_t}}x))ds \ .$$

Passing to weak*-limit we obtain $\rho_a^\kappa(x)$.

Lemma 3.5. *Let ω and $t > 0$ such that $t < \mathbf{t}_a(\omega)$ and $b = B_t(\omega) > 0$. Then*

$$\Big(\hat{E}_t \langle \rho_a^\kappa x, \rho_a^\kappa x \rangle_\infty - \langle \rho_a^\kappa x, \rho_a^\kappa x \rangle_t \Big)(\omega) \ \le \ c(\kappa) \hat{E}_t \Big(\rho_a \big(\int_0^\infty P_s(\hat{\Gamma}(P_s x, P_s x) \min(s, b) ds \big) \Big)(\omega)$$

holds for $\kappa > 1$. For $\kappa = 1$ and $0 < \beta < 1$

$$\Big(\hat{E}_t \langle \rho_a x, \rho_a x \rangle_\infty - \langle \rho_a x, \rho_a x \rangle_t \Big)(\omega) \ \le \ \frac{c}{\beta(1-\beta)^3} \pi_{t\wedge \mathbf{t}_a} \big(\int_0^\infty P_{\beta b + s} \hat{\Gamma}(P_s x, P_s x) \min(s, b) ds \big)(\omega).$$

Proof. Our starting point is

$$\langle \rho_a^\kappa(x), \rho_a^\kappa(x) \rangle_t \ = \ \int_0^{\mathbf{t}_a \wedge t} \hat{\pi}_s(\hat{\Gamma}(P^\kappa x, P^\kappa x))ds \ .$$

For the little bmo norm this implies

$$\hat{E}_t \langle \rho_a^\kappa x, \rho_a^\kappa x \rangle_\infty - \langle \rho_a^\kappa x, \rho_a^\kappa x \rangle_t \ = \ \hat{E}_t \int_{t\wedge \mathbf{t}_a}^{\mathbf{t}_a} \hat{\pi}_s(\hat{\Gamma}(P^\kappa x, P^\kappa x))ds \ .$$

Thus for $t > \mathbf{t}_a(\omega)$ we have 0. Let us assume $t < \mathbf{t}_a(\omega)$ and $b = B_t(\omega) > 0$. Then we observe that

$$\mathbb{E}(E_t \int_t^{\mathbf{t}_a} \pi_s(\hat{\Gamma}(P_{B_s}^\kappa x, P_{B_s}^\kappa x))ds | B_t = b)(\omega) \ = \ \pi_t \Big(\mathbb{E} \int_0^{\mathbf{t}_b} T_s(\hat{\Gamma}(P_{\tilde{B}_s}^\kappa x, P_{\tilde{B}_s}^\kappa x))ds \Big) \ .$$

On the right hand side we used the notation $\tilde{B}_s$ for a Brownian motion starting at b and $\mathbf{t}_b$ is the stopping time at 0. Let us fix $y = P_b x$. We use a well-known formula for local times (see [Bak85b, Formula (11)])

$$\mathbb{E} \int_0^{\mathbf{t}_b} f(t, \tilde{B}_t)dt \ = \ \frac{1}{2} \int_0^\infty \int_{|b-s|}^{b+s} \int_0^\infty f(r, s) d\mu_t(r) dt ds \tag{3.2}$$

where $\int_0^\infty e^{-\lambda r} d\mu_t(r) = e^{-t\sqrt{\lambda}}$. This implies

$$\mathbb{E}\int_0^{\mathbf{t}_b} T_s(\hat{\Gamma}(P^\kappa_{\tilde{B}_s}x, P^\kappa_{\tilde{B}_s}x))ds \;=\; \frac{1}{2}\int_0^\infty \int_{|b-s|}^{b+s} P_t\hat{\Gamma}(P_{\kappa s}x, P_{\kappa s}x)dtds \tag{3.3}$$

For $\kappa > 1$, let $\alpha = \frac{\kappa+1}{2\kappa}$. Then we observe with $\Gamma^2 \;\geq\; 0$ and monotonicity from Proposition 1.3 that

$$\begin{aligned}
\int_{|b-s|}^{b+s} P_t\hat{\Gamma}(P_{\kappa s}x, P_{\kappa s}x)dt \;&\leq\; \int_{|b-s|}^{b+s} P_{t+\alpha\kappa s}\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x)dt \\
&\leq \left(\int_{|b-s|}^{b+s}(t+\alpha\kappa s)dt\right)\frac{P_{|b-s|+\alpha\kappa s}(\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x))}{|b-s|+\alpha\kappa s} \\
&= \frac{2bs+\alpha\kappa s(b+s-|b-s|)}{|b-s|+\alpha\kappa s}P_{|b-s|+\alpha\kappa s}(\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x))\;.
\end{aligned}$$

Note that $b+s-|b-s| = 2\min(b,s)$. For $s \;\geq\; b$ we use monotonicity again and get

$$\begin{aligned}
&\frac{2(1+\alpha\kappa)bs}{|b-s|+\alpha\kappa s}P_{s-b+\alpha\kappa s}\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x) \\
&\leq \frac{2(1+\alpha\kappa)bs}{b+(\alpha\kappa-1)s}P_{b+(\alpha\kappa-1)s}\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x) \\
&\leq b\frac{2(1+\alpha\kappa)}{(\alpha\kappa-1)}P_{b+(\alpha\kappa-1)s}\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x))\;.
\end{aligned}$$

Note that for $b > s$ we have $|b-s|+\alpha\kappa s = b+(\alpha\kappa-1)s$ and hence

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty \int_{|b-s|}^{b+s} P_t\hat{\Gamma}(P_{\kappa s}x, P_{\kappa s}x)dtds \\
&\leq\; \max((1+\alpha\kappa), \frac{1+\alpha\kappa}{\alpha\kappa-1})\int_0^\infty P_{b+(\alpha\kappa-1)s}\hat{\Gamma}(P_{(1-\alpha)\kappa s}x, P_{(1-\alpha)\kappa s}x)\min(b,s)ds \\
&=\; 2\max(\frac{\kappa+3}{2}, \frac{3+\kappa}{\kappa-1})\int_0^\infty P_{b+s}\hat{\Gamma}(P_sx, P_sx)\min(b, \frac{2s}{\kappa-1})\frac{ds}{\kappa-1}\;.
\end{aligned}$$

We deduce the first assertion. For $\kappa = 1, \beta < 1$, let $\alpha = \frac{\beta+1}{2}$ and $\gamma = \frac{1-\beta}{2}$. Then, for $b > s$, $b-s+\alpha s \;\geq\; \beta b+\gamma s$ and hence

$$\begin{aligned}
&\frac{2bs+\alpha\kappa s(b+s-|b-s|)}{|b-s|+\alpha s}P_{|b-s|+\alpha s}(\hat{\Gamma}(P_{(1-\alpha)s}x, P_{(1-\alpha)s}x)) \\
&\leq\; \frac{2s(b+\alpha s)}{\beta b+\gamma s}P_{\beta b+\gamma s}(\hat{\Gamma}(P_{(1-\alpha)s}x, P_{(1-\alpha)s}x)) \;\leq\; \frac{4s}{\beta}P_{\beta b+\gamma s}(\hat{\Gamma}(P_{\gamma s}x, P_{\gamma s}x))\;.
\end{aligned}$$

For $s \;\geq\; b$ we have $s-b+\alpha s \;\geq\; \beta b+\gamma s$ and hence

$$\begin{aligned}
&\frac{2(1+\alpha)bs}{s-b+\alpha s}P_{s-b+\alpha s}(\hat{\Gamma}(P_{(1-\alpha)s}x, P_{(1-\alpha)s}x)) \\
&\leq\; \frac{2(1+\alpha)bs}{\beta b+\gamma s}P_{\beta b+\gamma s}(\hat{\Gamma}(P_{(1-\alpha)s}x, P_{(1-\alpha)s}x)) \;\leq\; 2b\frac{1+\alpha}{\gamma}P_{\beta b+\gamma s}(\hat{\Gamma}(P_{(1-\alpha)s}x, P_{(1-\alpha)s}x)) \\
&\leq\; \frac{8b}{1-\beta}P_{\beta b+\gamma s}(\hat{\Gamma}(P_{\gamma s}x, P_{\gamma s}x))\;.
\end{aligned}$$

We deduce the assertion from a change of variables which leads to $c(1-\beta)^{-3}$ or $c(1-\beta)^{-2}\beta^{-1}$. ■

The lower estimate in the following result is well-known in the commutative theory.

Theorem 3.6. *Let $x \in \mathcal{A}$ and $2 \leq p < \infty$ and $\Gamma^2 \geq 0$ and $\kappa \geq 1$. Then*

$$\|(\int_0^\infty P_s\hat{\Gamma}(P_{\kappa s}x, P_{\kappa s}x)\min(s,a)ds)^{1/2}\|_p \leq \|\rho_a^\kappa x\|_{h_p^c} .$$

For $\kappa > 1$.

$$\|\rho_a^\kappa x\|_{h_p^c} \leq c_p(\kappa) \|(\int_0^\infty P_s\hat{\Gamma}(P_s x, P_s x)sds)^{1/2}\|_p .$$

Proof. For the lower estimate, we calculate the conditional expectation of the square function onto $\rho_a(N)$. Indeed, let $y \in N$ then

$$\begin{aligned}
&\mathbb{E}\tau(\rho_a(y)^* \int_0^{\mathbf{t}_a} \hat{\pi}_s(\hat{\Gamma}(P^\kappa x, P^\kappa x))ds) = \mathbb{E}\int_0^{\mathbf{t}_a} \tau(\hat{E}_s(\rho_a(y^*))\hat{\pi}_s(\hat{\Gamma}(P^\kappa x, P^\kappa x)))ds \\
&= \mathbb{E}\int_0^{\mathbf{t}_a} \tau(\pi_s(P_{B_s}(y^*)\pi_s(\hat{\Gamma}(P_{\kappa B_s}x, P_{\kappa B_s}x)))ds = \mathbb{E}\int_0^{\mathbf{t}_a} \tau(y^* P_{B_s}\hat{\Gamma}(P_{\kappa B_s}x, P_{\kappa B_s}x))ds \\
&= \int_0^\infty \tau(y^* P_s\hat{\Gamma}(P_{\kappa s}x, P_{\kappa s}x))\min(a,s)ds .
\end{aligned}$$

For the upper estimate we note that the L_p^cMO is given by

$$\|\rho_a^\kappa x\|_{L_p^c mo} = \|\sup_t \hat{E}_t\langle\rho_a^\kappa x, \rho_a^\kappa x\rangle_\infty - \langle\rho_a^\kappa x, \rho_a^\kappa x\rangle_t\|_{p/2}^{1/2} .$$

We may assume $z = \int_0^\infty P_s\hat{\Gamma}(P_s x, P_s x)sds \in L_{p/2}(N)$. By Doob's inequality we find a $y \in L_{p/2}$ such that

$$\pi_{\mathbf{t}_a \wedge t}(P_{B_t}z) = \hat{E}_t(\rho_a(z)) \leq y$$

for all $t \geq 0$ and

$$\|y\|_{p/2} \leq c_{p/2}\|\rho_a(z)\|_{p/2} = c_{p/2}\|z\|_{p/2} .$$

With Lemma 3.5 we deduce that

$$\hat{E}_t\langle\rho_a^\kappa x, \rho_a^\kappa x\rangle_\infty - \langle\rho_a^\kappa x, \rho_a^\kappa x\rangle_t \leq c(\kappa)\hat{\pi}_{t\wedge\mathbf{t}_a}(Pz) \leq c(\kappa)y$$

for all $t \geq 0$. This implies the upper estimate. ∎

Remark 3.7. In the semi-commutative case where $P_t = P_t^{\mathbb{R}^n} \otimes id$ is the Poisson semigroup on $\mathbb{R}^n$ we have the estimate

$$\frac{P_{\beta t}}{\beta t} \leq \beta^{-n}\frac{P_t}{t}$$

which follows from the explicit representation as a convolution kernel. Choosing $\beta = (1 - \frac{1}{n})$ in Lemma 3.5, we obtain a polynomial estimate

$$\|\rho_a(x)\|_{h_p^c} \leq cn^3\|x\|_{H_p^c} .$$

Corollary 3.8. *Let $2 \leq p < \infty$ and $\Gamma^2 \geq 0$. Then*

$$\|(\int_0^\infty P_s\hat{\Gamma}_s(P_s x, P_s)sds)^{\frac{1}{2}}\|_p \sim_{c(p,\kappa)} \lim_{a\to\infty} \|\rho_a^\kappa(x)\|_{h_p^c} .$$

holds for all $\kappa > 1$.

Lemma 3.9. *Let $x \in \mathcal{A}$ such that*

$$\sup_s \|A((P_s x)^2)\|_2 < \infty .$$

Then the martingale $x_t = \hat{E}_t(\rho_a(x))$ has continuous path. Moreover, if $\Gamma^2 \geq 0$, then every martingale $\rho_a(x)$ with $x \in \mathcal{A}$ has continuous path.

Proof. Let us assume x selfadjoint (for convenience). We follow Lemma 2.1 and observe that the 4-norm satisfies

$$\begin{aligned}\|x_t - x_s\|_4^4 &= \tau(x_t^4) + \tau(x_s^4) - 4\tau(x_t^3 x_s) - 4\tau(x_s^3 x_t) + 4\tau(x_t^2 x_s^2) + 2\tau(x_t x_s x_t x_s) \\ &\le \tau(x_t^4) - 3\tau(x_s^4) - 4\tau(x_t^3 x_s) + 6\tau(x_t^2 x_s^2) \\ &= \tau(x_t^4) - \tau(x_s^4) - 4(\tau(x_t^3 x_s) - \tau(x_s^4)) + 6(\tau(x_t^2 x_s^2) - \tau(x_s^4)) \,.\end{aligned}$$

We note that

$$\tau(x_t^4) \;=\; \mathbb{E}\tau(\pi_{\mathbf{t}_a\wedge t}(P_{B_{\mathbf{t}_a\wedge t}}(x))^4)) \;=\; \mathbb{E}\tau((P_{B_{\mathbf{t}_a\wedge t}} x)^4) \,.$$

We use the Itô formula for $f(s) = (P_s x)^4$ and obtain

$$\begin{aligned}&(P_{B_{\mathbf{t}_a\wedge t}} x)^4 \;=\; (P_{B_{\mathbf{t}_a\wedge s}} x)^4 \\ &+ \int_{\mathbf{t}_a\wedge s}^{\mathbf{t}_a\wedge t} (P'_{B_r} x (P_{B_r} x)^3 + P_{B_r} x P'_{B_r} x (P_{B_r} x)^2 + (P_{B_r} x)^2 P'_{B_r} x P_{B_r} x + (P_{B_r} x)^3 P'_{B_r} x) dB_r + \\ &\int_{\mathbf{t}_a\wedge s}^{\mathbf{t}_a\wedge t} \big(P''_{B_r} x (P_{B_r} x)^3 + P'_{B_r} x P'_{B_r} x (P_{B_r} x)^2 + P'_{B_r} x P_{B_r} x P'_{B_r} x P_{B_r} x + P'_{B_r} x (P_{B_r} x)^2 P'_{B_r} x\big) dr + ..\end{aligned}$$

Indeed, every term in the second line yields four terms for the second derivative in the next line. Taking the expectation it suffices to estimate the terms with the second derivative (using the unusual normalization dr instead of $\frac{1}{2}dr$). Thus a uniform bound for $A^{1/2}x$ in the 4 norm implies that

$$|\tau(x_t^4) - \tau(x_s^4)| \;\le\; C\mathbb{E}|\mathbf{t}_a\wedge t - \mathbf{t}_a\wedge s| \;\le\; C|t-s| \max\{\|Ax\|_4^2 \|x\|_4^2 + \|A^2 x\|_4 \|x\|_4^3\} \,.$$

For the second term we observe that

$$\tau(x_t^3 x_s) \;=\; \mathbb{E}\tau\big(\pi_{\mathbf{t}_a\wedge t}((P_{B_{\mathbf{t}_a\wedge t}} x)^3)\pi_{\mathbf{t}_a\wedge s}(P_{B_{\mathbf{t}_a\wedge s}} x)\big) \;=\; \mathbb{E}\tau\big((P_{B_{\mathbf{t}_a\wedge t}} x)^3 T_{\mathbf{t}_a\wedge t - \mathbf{t}_a\wedge s} P_{B_{\mathbf{t}_a\wedge s}} x\big) \,.$$

We have to invoke the Itô formula for

$$\begin{aligned}&(P_{B_{\mathbf{t}_a\wedge t}} x)^3 \;=\; (P_{B_{\mathbf{t}_a\wedge s}} x)^3 \\ &+ \int_{\mathbf{t}_a\wedge s}^{\mathbf{t}_a\wedge t} (P'_{B_r} x (P_{B_r} x)^2 + P_{B_r} x P'_{B_r} x P_{B_r} x + (P_{B_r} x)^2 P'_{B_r} x) dB_r \\ &+ \int_{\mathbf{t}_a\wedge s}^{\mathbf{t}_a\wedge t} \big(P''_{B_r} x (P_{B_r} x)^2 + P'_{B_r} x P'_{B_r} x P_{B_r} x + P'_{B_r} x P_{B_r} x P'_{B_r} x\big) dr + ..\end{aligned}$$

Here we use P''_s for $\frac{d^2}{ds^2} P_s$. The first term vanishes again due to the martingale property. Then we use

$$\|T_r P_{B_{\mathbf{t}_a\wedge s}} x - P_{B_{\mathbf{t}_a\wedge s}} x\|_4 \;\le\; \|T_r x - x\|_4 \;=\; \|\int_0^r T_s A x\|_4 \;\le\; r\|Ax\|_4$$

in

$$\begin{aligned}&\mathbb{E}\tau\big((P_{B_{\mathbf{t}_a\wedge t}} x)^3 T_{\mathbf{t}_a\wedge t - \mathbf{t}_a\wedge s} P_{B_{\mathbf{t}_a\wedge s}} x\big) - \mathbb{E}\tau\big((P_{B_{\mathbf{t}_a\wedge s}} x)^4\big) \\ &= \mathbb{E}\tau\big((P_{B_{\mathbf{t}_a\wedge t}} x)^3 P_{B_{\mathbf{t}_a\wedge s}} x\big) - \mathbb{E}\tau\big((P_{B_{\mathbf{t}_a\wedge s}} x)^4\big) + \tau\big((P_{B_{\mathbf{t}_a\wedge t}} x)^3 (T_{\mathbf{t}_a\wedge t - \mathbf{t}_a\wedge s} - I) P_{B_{\mathbf{t}_a\wedge s}} x\big) \,.\end{aligned}$$

Applying Itô's formula for $P_{B_{\mathbf{t}_a\wedge t}} x$ we find an estimate of the form

$$|\tau(x_t^3 x_s) - \tau(x_s^4)| \;\le\; |t-s|(\|x\|_4^3 \|A^2 x\|_4 + \|x\|_4^3 \|Ax\|_4)$$

for $A^2 x, Ax \in L_4$. For the last term we have

$$\tau(x_t^2 x_s^2) \;=\; \mathbb{E}\tau((P_{B_{\mathbf{t}_a\wedge t}} x)^2 T_{\mathbf{t}_a\wedge t - \mathbf{t}_a\wedge s} (P_{B_{\mathbf{t}_a\wedge s}} x)^2) \,.$$

The Itô formula for $(P_{B_{\mathbf{t}_a\wedge t}} x)^2$ is simpler than above. According to our assumption

$$\|(T_{\mathbf{t}_a\wedge t - \mathbf{t}_a\wedge s} - I)((P_{B_{\mathbf{t}_a\wedge s}} x)^2)\|_2 \;\le\; C|\mathbf{t}_a\wedge t - \mathbf{t}_a\wedge s| \;\le\; C|t-s| \,.$$

Therefore the martingale satisfies the assumption of the noncommutative Kolmogorov theorem due to [GŁ85]. Finally, let us assume that $\Gamma^2 \ \geq \ 0$. Then we use $\hat{A}Px = 0$ and conlcude for selfadjoint x that

$$\begin{aligned} A(P_sxP_sx) &= \hat{A}(P_sxP_sx) + \frac{d}{ds}(P_sx)^2 \ = \ 2\hat{\Gamma}(P_sx, P_sx) + P_sxP_s'x + P_s'xP_sx \\ &= 2\Gamma(P_sx, P_sx) + 2P_s'xP_s'x + P_sxP_s'x + P_s'xP_sx \ . \end{aligned}$$

This implies with $0 \leq \Gamma(P_sx, P_sx) \leq P_s\Gamma(x, x)$ and Hölder's inequality that

$$\|A(P_sxP_sx)\|_2 \leq 2\|\Gamma(x, x)\|_2 + 2\|A^{1/2}x\|_4^2 + 2\|x\|_4\|A^{1/2}x\|_4 \ .$$

Thus for $x \in \mathcal{A}$, we have $\Gamma(x, x) \in \mathcal{A}$ and hence a uniform estimate in s. ■

The main advantage of the probabilistic model is that it allows to consider time and space derivatives simultaneously. Let us recall that in the space h_p^c, $1 \leq p < \infty$, we have an orthogonal projection P^{br} on the space of martingales

$$h_p^{br} \ = \ \{\int x_s dB_s \ : \ (x_s) \text{ adapted}\}$$

Of course, we have to read $\int x_s dB_s$ as a stochastic integral approximated by $\sum_{s_j} x_{s_j}(B_{s_{j+1}} - B_{s_j})$. We refer to the classical literature for approximation of the stopped process

$$(\int x_s dB_s)_{\mathbf{t}_a} \ = \ \int_0^{\mathbf{t}_a} x_s dB_s$$

which remains in h_p^{br}. Let us consider a martingale $z_t \in L_\infty(\Omega)\bar{\otimes}N$. Then the brownian projection z_t is the unique martingale $b_t \in h_p^{br}$ such that

$$\langle b, \int x_s dB_s\rangle_t \ = \ \langle z, \int x_s dB_s\rangle_t$$

holds for every adapted process x. Let us consider for example the simple tensor $z = f \otimes y$. We may assume

$$E_s^B(f) \ = \ \int_0^s g(r)dB_r$$

Let $z_s = \hat{E}_s(z)$. Note that

$$\begin{aligned} z_{s+h} - z_s &= E_{s+h}^B(f)y_{s+h} - E_s^B(f) \otimes y_s \\ &= (E_{s+h}^B(f) - E_s^B(f))y_s + E_s^B(f)(y_{s+h} - y_s) + (E_{s+h}^B(f) - E_s^B(f))(y_{s+h} - y_s) \ . \end{aligned} \tag{3.4}$$

Thus for $m = x_s(B_{s+h} - B_s)$ we find

$$\begin{aligned} &\hat{E}_s((m_{s+h}^* - m_s^*)(z_{s+h} - z_s)) \ = \ x_s^*\hat{E}_s((B_{s+h} - B_s)(z_{s+h} - z_s)) \\ &= x_s^*\hat{E}_s((B_{s+h} - B_s)(E_{s+h}^B(f) - E_s^B(f))y_s + x_s^*\hat{E}_s((B_{s+h} - B_s)E_s^B(f)(y_{s+h} - y_s)) \\ &\quad + x_s^*\hat{E}_s((B_{s+h} - B_s)(E_{s+h}^B(f) - E_s^B(f))(y_{s+h} - y_s)) \\ &= x_s^* \int_s^{s+h} g(r)dr \ y_s \ . \end{aligned}$$

Indeed, for the two additional terms we use commutativity and $\hat{E}_s \ = \ E_s^B \otimes E_s$. This yields 0 in both cases. Thus in general we find

$$\langle \int a_s dB_s, z\rangle_t \ = \ \int_0^t g(r)E_r(y)dr \quad \text{and} \quad b_t \ = \ \int_0^t g(r)E_r(y)dB_r \ .$$

This shows us how to extend the projection P^{br} by linearity. Since this procedure is less known in the non-commutative context we shall also show continuity with respect to the h_p norm. We come back to (3.4) and observe as above with the help of orthogonality that

$$\begin{aligned}\hat{E}_s((z_{s+h}-z_s)^*(z_{s+h}-z_s)) &= \hat{E}_s(((E^B_{s+h}(f)-E^B_s(f))y_s)^*(E^B_{s+h}(f)-E^B_s(f))y_s)\\ &+ \hat{E}_s((E^B_s(f)(y_{s+h}-y_s))^*E^B_s(f)(y_{s+h}-y_s))\\ &+ \hat{E}_s(((E^B_{s+h}(f)-E^B_s(f))(E^B_{s+h}(f)-E^B_s(f))(y_{s+h}-y_s))^*(y_{s+h}-y_s)) \,.\end{aligned}$$

For the last term we get

$$\begin{aligned}&\hat{E}_s(((E^B_{s+h}(f)-E^B_s(f))(E^B_{s+h}(f)-E^B_s(f))(y_{s+h}-y_s))^*(y_{s+h}-y_s))\\ &= \int_s^{s+h} g(r)dr\; E_s((y_{s+h}-y_s)^*(y_{s+h}-y_s)) \,.\end{aligned}$$

However, the Burkolder inequality implies that

$$\|\sum_j E_{s_j}(|y_{s_{j+1}}-y_{s_j}|^2)\|_{p/2} \;\le\; c(p)\|y\|_p^2 \,.$$

Thus for bounded g, the last term vanishes as long as the mesh size of the partition goes to 0. This yields

$$\langle z,z\rangle_t \;=\; \langle b,b\rangle_t + \langle \int_0^\infty E_s(f)dy_s, \int_0^\infty E_s(f)dy_s\rangle_t \,.$$

By approximation and linearity we deduce that

$$\langle P^{br}(z),P^{br}(z)\rangle_t + \langle (I-P^{br})(z),(I-P^{br})(z)\rangle_t \;=\; \langle z,z\rangle_t \,. \tag{3.5}$$

Lemma 3.10. *Let $1<p<\infty$. Then P^{br} and $(I-P^{br})$ are bounded, selfadjoint preserving maps on $L_p(\hat{M})$.*

Proof. By duality it suffices to consider $2\le p<\infty$. We see that on a dense set of martingales of the form $z=\sum_j f_j\otimes y_j$, the images $P^{br}(z)$ have continuous path and satisfy

$$\langle P^{br}(z),P^{br}(z)\rangle_t \;\le\; \langle z,z\rangle_t \,.$$

Thus the Burkholder-inequalities imply that

$$\|P^{br}(z)\|_{h_p^c} \;\le\; \|z\|_{h_p^c} \;\le\; c(p)\|z\|_p \,.$$

Note that that $P^{br}(z^*)=P^{br}(z)^*$. Since $P^{br}(z)$ has continuous path we deduce from [JK] that

$$\|P^{br}(z)\|_{L_p} \;\le\; c_1(p)\;\|z\|_{h_p} \;\le\; c_1(p)c(p)\|z\|_p \,.$$

The assertion follows by density. Moreover, the jump parts of z are mapped to $(I-P^{br})(z)$. ■

Lemma 3.11. *Let $x\in N$, then*

i) $\langle P^{br}\rho_a(x),P^{br}\rho_a(x)\rangle_t \;=\; 2\int_0^{\mathbf{t}_a\wedge t}\hat{\pi}_r(|P'x|^2)dr$;

ii) $\langle (Id-P^{br})\rho_a(x),(Id-P^{br})\rho_a(x)\rangle_t \;=\; 2\int_0^{\mathbf{t}_a\wedge t}\hat{\pi}_r(\Gamma(Px,Px))dr$.

Proof. According to Proposition 3.3 and (3.5)it suffices to show that

$$\langle P^{br}(\pi_{\mathbf{t}_a}(x)),P^{br}(\pi_{\mathbf{t}_a}(x))\rangle \;=\; 2\int_0^{t\wedge \mathbf{t}_a}\hat{\pi}_s(|P'x|^2)ds \,. \tag{3.6}$$

We deduce from

$$\langle n_{\mathbf{t}},m_{\mathbf{t}}\rangle_s \;=\; \langle n,m\rangle_{s\wedge \mathbf{t}}$$

that P^{br} commutes with stopping times. Therefore it suffices to consider the martingale $m_t(x) = \hat{\pi}_t(Px)+\int_0^t\hat{\pi}_s(\hat{A}Px)ds=\hat{\pi}_t(Px)$ and calculate the component corresponding to the brownian

motion. By approximation it suffices to consider $n = (y \otimes f)(B_{s+h} - B_s)$ such that f is a Σ_s-measurable bounded function. Let $F_t : M \to N$ be the conditional expectation corresponding to the trace preserving map π_t and $\tilde{y} = F_{s+h}(y) \in N$. Let μ be the spectral measure such that

$$\tau(\tilde{y} f(A) x) \;=\; \int_0^\infty f(\lambda) d\mu(\lambda) \,.$$

Then we have

$$\begin{aligned}&\mathbb{E}\tau((y \otimes f)(B_{s+h} - B_s)\pi_{s+h}(P_{B_{s+h}} x)) \\ &= \mathbb{E}(B_{s+h} - B_s) f \tau_N(F_{s+h}(y) P_{B_{s+h}} x) \;=\; \int_0^\infty \mathbb{E}(B_{s+h} - B_s) f e^{-\sqrt{\lambda} B_{s+h}} d\mu(\lambda) \,.\end{aligned}$$

To be more precise, we replace $g(t) = P_t x$ by a function $h(-\sqrt{A} t)x$ such that $h(z) = e^z v_\varepsilon(z)$ such that $v_\varepsilon(z)$ vanishes at 0 and converges to 1 as ε goes to 0. Using a stopping time $\mathbf{t}_a^\delta$ which stops the brownian motion at $\delta > 0$ this calculation can be justified. By Itô's formula we have

$$e^{-\sqrt{\lambda} B_{s+h}} \;=\; e^{-\sqrt{\lambda} B_s} - \sqrt{\lambda} \int_s^{s+h} e^{-\sqrt{\lambda} B_r} dB_r + \lambda \int_s^{s+h} e^{-\sqrt{\lambda} B_r} dr \,.$$

This yields

$$\begin{aligned}&\mathbb{E}\tau((y \otimes f)(B_{s+h} - B_s)\pi_{s+h}(P_{B_{s+h}} x)) \;=\; \mathbb{E}\tau((y \otimes f)(B_{s+h} - B_s)\pi_{s+h}(P_{B_s}(x)) \\ &+ \int_s^{s+h} \mathbb{E}\tau((y \otimes f)\pi_{s+h}(P'_{B_r} x))\, 2dr + \int_s^{s+h} \mathbb{E}\tau((y \otimes f)(B_{s+h} - B_s)\pi_{s+h}(P''_{B_r} x))dr \\ &= \int_s^{s+h} \mathbb{E}\tau((y \otimes f)\pi_{s+h}(P'_{B_r} x))dr \\ &+ \int_s^{s+h} \mathbb{E}\tau((y \otimes f)(B_{s+h} - B_r)\pi_{s+h}(P''_{B_r} x)) + \int_s^{s+h} \mathbb{E}\tau((y \otimes f)(B_r - B_s)\pi_{s+h}(P''_{B_r} x))dr \,.\end{aligned}$$

By independence the last two terms are 0. Note that

$$\|\pi_{s+h}(y) - \pi_r(y)\|_2^2 = 2(\tau(y^* y) - \tau(T_{s+h-r}(y)^* y)) \;\le\; (s+h-r)\|Ay\|_2 \|y\|_2$$

implies

$$1_{B_r > 0} \|\pi_{s+h}(P'_{B_r} x) - \pi_r(P'_{B_r} y)\|_2^2 \;\le\; (s+h-r)\|A^2 y\|_2 \|Ay\|_2 \,.$$

Thus by continuity, we obtain

$$b_t \;=\; \int_0^t \pi_r(P'_{B_r} x) dB_r \,.$$

Hence for the bracket, we deduce (3.6) (with $d\langle B_r, B_r\rangle = 2dr$). ∎

Lemma 3.12. *Let $x, y \in L_2$. Let Pr be the projection onto $(\ker(A))^\perp$. Then*

$$\lim_{a\to\infty} \tau(\langle (I - P^{br})\rho_a(x), (I - P^{br})\rho_a(y)\rangle_{\mathbf{t}_a}) \;=\; \frac{1}{4}\tau((I - \mathrm{Pr})(x)^*(I - \mathrm{Pr})(y)) \,.$$

Proof. By polarization we have

$$\langle (Id - P^{br})\pi_{\mathbf{t}_a}(x), (Id - P^{br})\pi_{\mathbf{t}_a}(y)\rangle_\infty \;=\; 2\int_0^{t\wedge \mathbf{t}_a} \hat{\pi}_s \Gamma(Px, Py) ds \,.$$

Thus taking the trace yields

$$\begin{aligned}&\tau(\langle (Id - P^{br})\pi_{\mathbf{t}_a}(x), (Id - P^{br})\pi_{\mathbf{t}_a}(y)\rangle_\infty) \\ &= 2\mathbb{E}\int_0^{\mathbf{t}_a} \tau(\hat{\pi}_s \Gamma(Px, Py)) ds \;=\; 2\mathbb{E}\int_0^{\mathbf{t}_a} \tau(P_{B_s} x^* A P_{B_s} y) ds \,.\end{aligned}$$

Again, we may use polarization and hence it suffices to establish the result for $x = y$. Let dE_λ be the spectral measure of A and $\omega_x(T) = (x, Tx)$. We use the well-known formula (see [Bak85b])

$$\mathbb{E}\int_0^{\mathbf{t}_a} f(B_s)ds \;=\; \int_0^\infty \min(a,s)f(s)ds\;. \tag{3.7}$$

This implies

$$\begin{aligned}\mathbb{E}\int_0^{\mathbf{t}_a} \tau(P_{B_s}x^*AP_{B_s}x)ds \;&=\; \mathbb{E}\int_0^{\mathbf{t}_a}(x, P_{2B_s}A(x))ds\\ =\int_0^\infty \mathbb{E}\int_0^{\mathbf{t}_a} e^{-2\sqrt{\lambda}B_s}\lambda ds\omega_x(dE_\lambda) \;&=\; \int_0^\infty\int_0^\infty \min(s,a)e^{-2\sqrt{\lambda}s}ds\omega_x(dE_\lambda)\;.\end{aligned}$$

For $\lim_{a\to\infty}$ we find

$$\lambda\int_0^\infty e^{-2\sqrt{\lambda}s}sds \;=\; \frac{1}{4}\int_0^\infty e^{-2\sqrt{\lambda}s}(2\sqrt{\lambda}s)^2\frac{ds}{s} \;=\; \frac{1}{4}$$

for $\lambda > 0$ and equals 0 else. Thus we get

$$\lim_{a\to\infty}\tau(\langle (Id-P^{br})\pi_{\mathbf{t}_a}(x), (Id-P^{br})\pi_{\mathbf{t}_a}(x)\rangle_\infty) \;=\; \frac{1}{2}\langle (I-\mathrm{Pr})(x), (I-\mathrm{Pr})(x)\rangle\;.$$

Thus the general formula is established by polarization. ∎

The next Lemma deals with Hardy spaces and follows closely Bakry's proof.

Lemma 3.13. *Assume that $\Gamma^2 \geq 0$ and (T_t) admits Markov dilation path. Then*

$$\sup_a \|(I-P^{br})\rho_a(A^{1/2}x)\|_{H_p^c} \;\leq\; c(p)\;\|\Gamma(x,x)^{\frac{1}{2}}\|_p\;.$$

holds for $2 < p < \infty$.

Proof. Let $x \in \mathcal{A}$. We consider the function

$$f(s) \;=\; \Gamma(P_sx, P_sx)$$

and claim that $y_t = \hat{\pi}_t(f)$ is a submartingale. Indeed, we know that

$$m_t(f) \;=\; \pi_t(f) + \int_0^t \hat{\pi}_r(\hat{A}f)dr$$

is a martingale. This implies that

$$E_s(\hat{\pi}_t(f)) \;=\; E_s(m_t(f)) - E_s(\int_0^t \hat{\pi}_r(\hat{A}f)dr) \;=\; m_s(f) - E_s(\int_0^t \hat{\pi}_r(\hat{A}f)dr)\;.$$

Let us calculate the right hand side:

$$\frac{\partial f}{\partial s}(s) \;=\; \Gamma(P_s'x, P_sx) + \Gamma(P_sx, P_s'x)$$

and hence

$$\begin{aligned}\frac{\partial^2 f}{\partial s^2}(s) &= \Gamma(P_s''x, P_sx) + \Gamma(P_sx, P_s''x) + 2\Gamma(P_s'x, P_s'x)\\ &= \Gamma(AP_sx, P_sx) + \Gamma(P_sx, AP_sx) + 2\Gamma(P_s'x, P_s'x)\;.\end{aligned}$$

Therefore we obtain

$$-\hat{A}(f) \;=\; (\frac{\partial^2}{\partial s^2} - A)(f) \;=\; 2\Gamma^2(P_sx, P_sx) + 2\Gamma(P_s'x, P_s'x)\;. \tag{3.8}$$

The same equation will allow us to estimate the increasing part of the bracket y_t defined as the limit of

$$\langle y\rangle_t \;=\; \lim_\sigma \sum_j E_{t_j}(y_{t_{j+1}} - y_{t_j}) \tag{3.9}$$

where the limit is taken along some ultrafilter on partitions of the interval $[0,t]$. Let $r_t = \int_0^t \hat{\pi}_s(\hat{A}f)ds$. Clearly, the bracket operation vanishes on the martingale part. We obtain

$$E_{t_j}(\int_{t_j}^{t_{j+1}} \hat{\pi}_s(\hat{A}f)ds) = \int_{t_j}^{t_{j+1}} \hat{\pi}_{t_j}(\hat{T}_{s-t_j}\hat{A}f)ds \;=\; \pi_{t_j}(f - \hat{T}_{t_{j+1}}f) \approx -(t_{j+1}-t_j)\pi_{t_j}(\hat{A}f)\,.$$

Thus by L_p continuity of $\hat{\pi}_s(\hat{A}f)$ we find

$$\langle y\rangle_t \;=\; \int_0^t \hat{\pi}_s(2\Gamma^2(P_sx,P_sx) + 2\Gamma(P_s'x,P_s'x))ds\,.$$

This does not change if we add stopping times, i.e. we have

$$\begin{aligned}\langle y_{\mathbf{t}_a}\rangle_t &= 2\int_0^{t\wedge\mathbf{t}_a} \hat{\pi}_s(\Gamma^2(P_sx,P_sx))ds + 2\int_0^{t\wedge\mathbf{t}_a} \hat{\pi}_s(\Gamma(P_s'x,P_s'x))ds \\ &\geq\; \langle (I-P^{br})(\pi_{\mathbf{t}_a}(A^{1/2}x)),(I-P^{br})(\pi_{\mathbf{t}_a}(A^{1/2}x))\rangle_t\,.\end{aligned}$$

According to Lemma 1.2 we find with $p/2>1$ that

$$\|\langle y\rangle_{\mathbf{t}_a}\|_{\frac{p}{2}} \;\leq\; c_p\|y_{\mathbf{t}_a}\|_{\frac{p}{2}}\,. \tag{3.10}$$

It is time to apply subharmonicity again. Now in the form

$$\Gamma(P_sx,P_sx) \;\leq\; P_s(\Gamma(x,x))\,.$$

Therefore

$$\hat{\pi}_{\mathbf{t}_a}(\Gamma(Px,Px)) \;\leq\; \hat{\pi}_{\mathbf{t}_a}(P\Gamma(x,x)) \;=\; \rho_a(\Gamma(x,x))\,.$$

Now, we note that $\pi_{\mathbf{t}_a}: N\to M$ is a trace preserving *-homomorphism. This implies

$$\|\rho_a(\Gamma(x,x))\|_{\frac{p}{2}} \;=\; \|\Gamma(x,x)\|_{\frac{p}{2}}\,.$$

We deduce that

$$\|(I-P^{br})\rho_a(A^{1/2}x)\|_{h_p^c} \;\leq\; c(p)\|\langle (I-P^{br})\rho_ax,(I-P^{br})\rho_ax\rangle\|_{p/2}^{1/2} \;\leq\; c'(p)\|\Gamma(x,x)^{\frac{1}{2}}\|_p\,.$$

We may replace the h_p^c norm by the H_p^c norm, because $\rho_a(x)$ and $P^{br}(\rho_a(x))$ have continuous path. Thus $I - P^{br}(\rho_a(x))$ also has continuous path. ■

We are now well-prepared for our main result on Riesz transforms.

Theorem 3.14. *Let T_t be a completely positive selfadjoint semigroup with negative generator A, which admits a Markov dilation and satisfies $\Gamma^2 \geq 0$. Let $2<p<\infty$. Then*

$$\|A^{\frac{1}{2}}x\|_p \;\leq\; c_p\;\max\{\|\Gamma(x,x)^{\frac{1}{2}}\|_p, \|\Gamma(x^*,x^*)^{\frac{1}{2}}\|_p\}$$

holds for all x.

Proof. Let $\delta>0$. Using the fact that Pr is a contraction, we may find $y_0\in L_{p'}(N)$ such that $\mathrm{Pr}(y_0)=0$, $\|y_0\|_{p'}\leq 1$ and

$$\|A^{1/2}x\|_p \;\leq\; 2(1+\delta)|\tau(y_0^*A^{1/2}x)|\,. \tag{3.11}$$

By approximation we may assume that $y_0\in L_2(N)$ and still satisfies (3.11). Since $\mathrm{rg}(A^{1/2}) = \ker(A^{1/2})^\perp = (\ker(A))^\perp$ we may approximate y_0 by $A^{1/2}y\in L_{p'}(N)$ such that $\|A^{1/2}y\|_{p'}\leq 1$ and

$$\|A^{1/2}x\|_p \;\leq\; 3|\tau(A^{1/2}y^*A^{1/2}x)|\,.$$

We fix $a>0$ and according to [JK] we decompose $\rho_a(A^{1/2}y) \;=\; m_c+m_r+m_d$ such that

$$\|m_c\|_{h^c_{p'}}+\|m_r\|_{h^r_{p'}}+\|m_d\|_{h^d_{p'}} \;\le\; c(p')\,\|\rho_a(A^{1/2}y)\|_{p'} \;\le\; 2c(p')\;.$$

Since $\rho_a(A^{1/2}x)$ has continuous path (see Lemma 3.9), we know that

$$\langle m_d^*,(I-P^{br})\rho_a(A^{1/2}x)\rangle \;=\; 0\;.$$

Therefore we obtain from Lemma 3.13 that

$$\begin{aligned}
&|\mathbb{E}\tau((I-P^{br})\rho_a(A^{1/2}y^*)(I-P^{br})(\rho_a(A^{1/2}x))|\\
&=|\mathbb{E}\tau((m_c^*+m_r^*+m_d^*)(I-P^{br})(\rho_a(A^{1/2}x))|\\
&=|\mathbb{E}\tau(((I-P^{br}(m_c)^*+(I-P^{br})(m_r^*))(I-P^{br})(\rho_a(A^{1/2}x))|\\
&=|\mathbb{E}\tau(\langle m_c,(I-P^{br})(\rho_a(A^{1/2}x))\rangle)|+|\overline{\mathbb{E}\tau(\langle m_r^*,(I-P^{br})(\rho_a(A^{1/2}x^*))\rangle)}|\\
&\le \|m_c\|_{h^c_{p'}}\|(I-P^{br})(\rho_a(A^{1/2}x))\|_{h^c_p}+\|m_r^*\|_{h^c_{p'}}\|(I-P^{br})(\rho_a(A^{1/2}x^*))\|_{h^c_p}\\
&\le c(p')c(p)\|\rho_a(A^{1/2}y)\|_{p'}(\|\Gamma(x,x)^{\frac12}\|_p+\|\Gamma(x^*,x^*)^{\frac12}\|_p)\;.
\end{aligned}$$

Note that $(I-\mathrm{Pr})(A^{1/2}x)=A^{1/2}x$. Therefore Lemma 3.12 shows that

$$\begin{aligned}
|\tau(A^{1/2}y^*A^{1/2}x)| &\le 4\lim_{a\to\infty}|\mathbb{E}\tau((I-P^{br})\rho_a(A^{1/2}y^*)(I-P^{br})(\rho_a(A^{1/2}x))|\\
&\le 8c(p')c(p)(\|\Gamma(x,x)^{\frac12}\|_p+\|\Gamma(x^*,x^*)^{\frac12}\|_p)\;.
\end{aligned}$$

By our choice of y we deduce the assertion. ■

As a further application we compare the martingale H_p-norms and the semigroup H_p-norms from [JLMX06].

Theorem 3.15. *Let $2\le p<\infty$ and $\Gamma^2 \;\ge\; 0$ and $\kappa>1$.*

i) *Then*

$$\|x\|_{H^c_p}\sim_{c(p)}\lim_a\|P^{br}\rho^\kappa_a(x)\|_{h^c_p}\sim_{\tilde c(p)}\lim_a\|\rho^\kappa_a(x)\|_{H^c_p}\;.$$

ii) *If moreover, the assumption of Lemma 2.4 or Lemma 2.6 is satisfied, then*

$$\|x\|_{H^c_p}\sim_{c(p)}\lim_a\|(I-P^{br})\rho^\kappa_a(x)\|_{h^c_p}\;.$$

and

$$\|A^{1/2}x\|_{H^c_p(P)} \;\le\; c(p)\|\Gamma(x,x)^{\frac12}\|_p\;.$$

Proof. Let us start with an easy application of Theorem 3.6, namely the condition $\Gamma^2 \;\ge\; 0$ implies

$$\begin{aligned}
&\|\int_0^\infty\hat\Gamma(P_sx,P_sx)s^2\frac{ds}{s}\|_{p/2}^{1/2} \;=\; (\kappa+1)\|\int_0^\infty\hat\Gamma(P_{(\kappa+1)s}x,P_{(\kappa+1)s}x)sds\|_{p/2}^{1/2}\\
&\le(\kappa+1)\;\|\int_0^\infty P_s\hat\Gamma(P_{\kappa s}x,P_{\kappa s}x)sds\|_{p/2}^{1/2}\\
&\le(\kappa+1)\;\lim_a\|\int_0^\infty P_s\hat\Gamma(P_{\kappa s}x,P_{\kappa s}x)\min(s,a)sds\|_{p/2}^{1/2} \;\le\; (\kappa+1)\lim_a\|\rho^\kappa_a(x)\|_{h^c_p}\;.
\end{aligned}$$

The converse is given by Theorem 3.6. The same argument in combination with Lemma 3.11 also shows that

$$\|\int_0^\infty|P'_sx|^2sds\|_{p/2}^{1/2}\sim_{c(p,\kappa)}\lim_a\|P^{br}\rho^\kappa_a(x)\|_{h^c_p}$$

and

$$\|\int_0^\infty \Gamma(P_sx, P_sx)sds\|_{p/2}^{1/2} \sim_{c(p,\kappa)} \lim_a \|(I - P^{br})\rho_a^\kappa(x)\|_{h_p^c} .$$

We refer to [Jun08] for

$$\|\int_0^\infty \Gamma(P_sx, P_sx)sds\|_{p/2}^{1/2} \;\le\; c(p)\|x\|_{H_p^c(P)} .$$

Assuming the condition of Lemma 2.6 for Γ or under the assumption of Lemma 2.4 we have

$$\|\int_0^\infty \Gamma(T_sx, T_sx)sds\|_{p/2}^{1/2} \;\le\; c(p)\|\int_0^\infty \Gamma(P_sx, P_sx)sds\|_{p/2}^{1/2} .$$

Thus Theorem 2.9 iii) yields the missing estimate in ii), because the $H_p^c(P)$ and $H_p^c(T)$ are comparable, see again [Jun08]. In that situation the last assertion follows from Lemma 3.13. ■

4. BMO spaces

In the recent years the theory of BMO spaces has been extended to semigroups of positive operators on $\mathbb{R}^n$ assuming that their kernels satisfy certain regularity conditions (see [DY05a]). In this part we compare different candidates for the BMO-norm. Our main motivation is the Garsia norm for the Poisson semigroup on the circle. In full generality we define for a semigroup T_t of completely positive maps the norm

$$\|x\|_{BMO_c(T)} \;=\; \sup_t \|T_t|x|^2 - |T_tx|^2\|_\infty^{1/2} .$$

Lemma 4.1. *Let (T_t) be a semigroup of completely positive maps on a von Neumann algebra. Then $BMO_c(T)$ defines a normed space. Moreover, if (T_t) has a reversed martingale dilation and $\Gamma^2 \;\ge\; 0$, then the reversed martingale $(\pi_s(T_s))$ satisfies*

$$\|\pi_0(x)\|_{bmo_c} \;=\; \|x\|_{BMO_c(T)} .$$

Proof. Let us fix $t > 0$ and define the homogenous expression $\|x\|_t = \|T_t|x|^2 - |T_tx|^2\|^{1/2}$. Let $d = T_t(1)$ and e the support projection of d. It is easy to see that $T_t(N) \subset eNe = M$. Then $\tilde{T}_t(x) = d^{-1/2}T_t(x)d^{-1/2}$ is a well-defined unital completely positive map $\tilde{T}_t : N \to M$. Let $N \otimes_{T_t} M$ be the Hilbert C^*-module over M with M inner product

$$\langle a \otimes b, c \otimes d\rangle \;=\; b^*\tilde{T}_t(a^*c)d .$$

Since $\tilde{T}_t$ is unital we obtain *-homomorphism $\pi : N \to \mathcal{L}(N \otimes_{\tilde{T}_t} M) \cong \mathrm{Mult}(K(\ell_2) \otimes M)$ such that

$$\tilde{T}_t(x) \;=\; e_{11}\pi(x)e_{11} .$$

This implies

$$T_t(x) \;=\; d^{1/2}e_{11}\pi(x)e_{11}d^{1/2} .$$

Therefore we get

$$\begin{aligned} T_t(x^*x) - T_t(x^*)T_t(x) \;&=\; d^{1/2}e_{11}\pi(x)^*\pi(x)e_{11}d^{1/2} - d^{1/2}e_{11}\pi(x)^*e_{11}d^{1/2}d^{1/2}e_{11}\pi(x)e_{11}d^{1/2} \\ &= d^{1/2}e_{11}\pi(x)^*(1 - e_{11}de_{11})\pi(x)e_{11}d^{1/2} . \end{aligned}$$

This implies that the linear map $u : N \to \mathcal{L}(N \otimes_{\tilde{T}_t} M)$ defined by

$$u(x) \;=\; (1 - e_{11}de_{11})^{\frac{1}{2}}\pi(x)e_{11}d^{1/2}$$

is an isometric embedding of N equipped with the norm $\| \, \|_t$. An alternative proof can be derived from (2.2)

$$T_t|x|^2 - |T_tx|^2 \;=\; 2\int_0^t T_{t-s}\Gamma(T_sx, T_sx)ds .$$

Thus the GNS construction for the positive form $T_{t-s}\Gamma$ allows us to find linear maps $u_{ts}: N \to C(N)$ such that

$$T_t|x|^2 - |T_tx|^2 \;=\; \int_0^t |u_{ts}(x)|^2 ds \;.$$

This provides an embedding in $L_2^c([0,t])\otimes_{\min} C(N)$. Now, we assume that T_t admits a martingale dilation as in section 1. We consider the martingale $m_s \;=\; \pi_s(T_sx) = E_{[s}(\pi_0(x))$ (see Lemma 2.1). The part ii) shows that

$$E_{[t}(\langle m,m\rangle_0 - \langle m,m\rangle_t) = 2E_{[t}\int_0^t \pi_s(\Gamma(T_sx,T_sx))ds \;=\; 2\pi_t\int_0^t T_{t-s}(\Gamma(T_sx,T_sx))ds$$
$$= \pi_t(T_t|x|^2 - |T_tx|^2) \;.$$

Taking the supremum over all t, we deduce the assertion. ■

The BMO norm for the probabilistic model is closely related to the associated Poisson semigroup.

Proposition 4.2. *Let (T_t) be a semigroup with a Markov dilation.*

i) *Let $x \in N$ and $a > 0$. Then*

$$\|x\|_{BMO_c(P)} \;=\; \|\rho_a(x)\|_{BMO_c} \;.$$

ii) $\frac{1}{90}\|P^{br}\rho_a(x)\|_{bmo_c} \;\le\; \sup_b \|\int_0^\infty P_{b+s}|P_s'|^2\min(s,b)ds\|^{\frac12} \;\le\; 2\|P^{br}\rho_a(x)\|_{bmo_c}$.

iii) *Assume* $\Gamma^2 \;\ge\; 0$. *Then*

$$\frac{1}{90}\|(I-P^{br})\rho_a(x)\|_{bmo_c} \le \sup_b \|\int_0^\infty P_{b+s}\Gamma(P_sx,P_sx)\min(s,b)ds\|^{\frac12} \le 2\|(I-P^{br})\rho_a(x)\|_{bmo_c}\,.$$

Proof. We recall that $\hat{E}_t(\rho_a(x)) = \hat{\pi}_{\mathbf{t}_a\wedge t}(Px)$ and $\rho_a(x) = \pi_{\mathbf{t}_a}(x)$. Hence we get

$$\hat{E}_t(|\rho_a(x)|^2) - |\hat{E}_t(\rho_a(x))|^2 = \pi_{\mathbf{t}_a\wedge t}(P_{B_{\mathbf{t}_a\wedge t}}|x|^2 - |P_{B_{\mathbf{t}_a\wedge t}}x|^2)\;,$$

for $\mathbf{t}_a(\omega) > t$. Thus in any case we have

$$\operatorname{ess\,sup}_\omega \|\hat{E}_t(|\rho_a(x)|^2) - |\hat{\pi}_{\mathbf{t}_a\wedge t}(Px)|^2\| \;\le\; \sup_s \|\pi_{\mathbf{t}_a\wedge t}(P_s|x|^2 - |P_sx|^2)\| \;\le\; \|x\|^2_{BMO_c(P)}\;.$$

However, for $t = 0$ we recall that $B_0(\omega) = a$ almost everywhere. This means $B_t = a + \tilde{B}_t$ where $\tilde{B}_t$ is a centered brownian motion. Since $\limsup_t |\tilde{B}_t|/\sqrt{2t\log\log t} \;=\; 1$, we know that with probability 1 the process $|\tilde{B}_t|$ exceeds a. Thus with probability 1 the process B_t hits 0 or $2a$. Hence with probability $\frac12$ the process hits $2a$ before it hits 0. Let us assume that $B_{t(\omega)}(\omega) = 2a$ and $B_s(\omega) > 0$ for $0 < s < t(\omega)$. By starting a new brownian motion at $t(\omega)$, we see with conditional probability $\frac12$ we have $B_{t'(\omega)} \;=\; 4a$ for some $t(\omega) < t'(\omega)$ and $B_s(\omega) > 0$ for all $t(\omega) < s < t'(\omega)$. By induction we deduce that with probability 2^{-n} the process B_t hits 2^na before it hits 0. Thus given any $b > 0$, we may choose n such that $2^na > b$. We see that with positive probability there exists $t_n(\omega)$ such that $B_{t_n(\omega)} = 2^na$ and $B_s(\omega) > 0$ on $[0,t_n(\omega)]$ and B_s is continuous. By continuity there exists $t(\omega) \in [t_n(\omega),\mathbf{t}_a(\omega)]$ such that $B_{t_\omega} = b$. In particular,

$$\|\hat{E}_{t(\omega)}(|\rho_a(x)|^2) - |\hat{\pi}_{\mathbf{t}_a\wedge t}(Px)|^2\| \;=\; \|\pi_{t(\omega)}(P_{B_{t(\omega)}}|x|^2 - |P_{B_{t(\omega)}}x|^2)\| \;=\; \|P_b|x|^2 - |P_bx|^2\|\;.$$

Taking the supremum over all b yields i). For the proof of iii) we first apply Lemma 3.11 and then Lemma 3.5. This immediately yields the first inequality (after a concise review of the involved constant for $\beta = \frac23$). For the upper estimate of this term, we recall that with positive probability

every value b is hit. Then we start in (3.3) for a fixed $b = B_t(\omega)$. We use the monotonicity $\frac{P_{b+s}(z)}{b+s} \le \frac{P_t(z)}{t}$ and find

$$\begin{aligned}\mathbb{E}\int_0^{\mathbf{t}_b} T_s(\Gamma(P_{\tilde{B}_s}x, P_{\tilde{B}_s}x))ds \;&=\; \frac{1}{2}\int_0^\infty \int_{|b-s|}^{b+s} P_t\Gamma(P_sx,P_sx)dtds\\ &\ge\; \frac{1}{2}\int_0^\infty \frac{P_{b+s}\Gamma(P_sx,P_sx)}{b+s}\left(\int_{|b-s|}^{b+s} tdt\right)ds \;=\; \int_0^\infty \frac{P_{b+s}\Gamma(P_sx,P_sx)}{b+s}bs\, ds\\ &\ge\; \frac{1}{2}\int_0^\infty P_{b+s}\Gamma(P_sx,P_sx)\min(b,s)ds\,.\end{aligned}$$

The proof of ii) is similar but we only need $|P_tz|^2 \le P_t|z|^2$ instead of $\Gamma^2 \ge 0$. ∎

Proposition 4.3. *Let (T_t) be a semigroup of completely positive selfadjoint maps and (P_t) the associated Poisson semigroup. Let $x \in \mathcal{A}$. Then*

i) $P_b|x|^2 - |P_bx|^2 = \int_0^\infty \int_{\max\{0,v-b\}}^b P_{b+2u-s}\hat{\Gamma}(P_sx,P_sx)duds$, *and, assuming* $\Gamma^2 \ge 0$,

$$\frac{1}{4}\int_0^\infty P_{b+s}\hat{\Gamma}(P_sx,P_sx)\min(s,b)ds \;\le\; P_b|x|^2-|P_bx|^2 \;\le\; 180\int_0^\infty P_{\frac{b}{3}+s}\hat{\Gamma}(P_sx,P_sx)\;\min(\frac{b}{3},s)ds\,;$$

ii) $\sup_b \| \int_0^\infty P_{b+s}|P_s'x|^2\min(s,b)ds\| \;\le\; 4\,\|x\|^2_{BMO_c(P)}$;

iii) $\sup_b \| \int_0^\infty P_{b+s}\Gamma(P_sx,P_sx)\min(s,b)ds\| \;\le\; 4\,\|x\|^2_{BMO_c(P)}$ *provided* $\Gamma^2 \ge 0$.

Proof. For the proof of i) we recall from (2.2) applied to P_t that

$$P_b|x|^2 - |P_bx|^2 \;=\; 2\int_0^b P_{b-s}\Gamma_{A^{1/2}}(P_sx,P_sx)ds\,.$$

We recall from [Jun08] that

$$\Gamma_{A^{1/2}}(y,y) \;=\; \int_0^\infty P_t\Gamma(P_ty,P_ty)dt + \int_0^\infty P_t|P_t'y|^2dt$$

holds for $y \in \mathcal{A}$. Combining these equations we obtain with a change of variables ($v = s+t$, $u = t$)

$$\begin{aligned}P_b|x|^2 - |P_bx|^2 \;&=\; 2\int_0^b\int_0^\infty P_{b-s+t}\hat{\Gamma}(P_{s+t}x,P_{s+t}x)dtds\\ &=\; 2\int_0^\infty\int_{\max\{0,v-b\}}^v P_{b-v+2u}\hat{\Gamma}(P_vx,P_vx)dudv \qquad (4.1)\end{aligned}$$

We apply $\hat{\Gamma}^2 \ge$ and monotonicity 1.3 and split the integral

$$\begin{aligned}&2\int_0^\infty\int_{\max\{0,v-b\}}^v P_{b-v+2u}\hat{\Gamma}(P_vx,P_vx)dudv\\ &\ge\; 2\int_0^\infty\int_{\max\{0,v-b\}}^v P_{b+2u}\hat{\Gamma}(P_{2v}x,P_{2v}x)dudv\\ &\ge\; 2\int_0^\infty\left(\int_{\max\{0,v-b\}}^v \frac{b+2u}{b+2v}du\right)P_{b+2v}\hat{\Gamma}(P_{2v}x,P_{2v}x)dv\\ &=\; \int_0^\infty \frac{2bv+4v^2-2b\max\{0,v-b\}-4\max\{0,v-b\}^2}{2(b+2v)}P_{b+2v}\hat{\Gamma}(P_{2v}x,P_{2v}x)dv\\ &\ge\; \int_0^b P_{b+2v}\hat{\Gamma}(P_{2v}x,P_{2v}x)vdv + \int_b^\infty \frac{4bv}{2(b+2v)}P_{b+2v}\hat{\Gamma}(P_{2v}x,P_{2v}x)dv\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{2}\int_0^b P_{b+2v}\hat{\Gamma}(P_{2v}x,P_{2v}x)2vdv+\frac{1}{2}b\int_b^\infty P_{b+2v}\hat{\Gamma}(P_{2v}x,P_{2v}x)dv\\
&\geq \frac{1}{2}\int_0^\infty P_{b+2v}\hat{\Gamma}(P_{2v}x,P_{2v}x)\min(2v,b)dv\ .
\end{aligned}
$$

Without $\Gamma^2 \geq 0$ we only obtain

$$P_b|x|^2-|P_bx|^2 \;\geq\; \frac{1}{2}\int_0^\infty P_{b+2v}|P'_{2v}x|^2\min(2v,b)dv \;=\; \frac{1}{4}\int_0^\infty P_{b+v}|P'_vx|^2\min(v,b)dv\ .$$

This yields iii) and iv). To complete the proof of i) we start with (4.1) and the Γ^2 condition:

$$
\begin{aligned}
&P_b|x|^2-|P_bx|^2 \;=\; 2\int_0^\infty\int_{\max\{0,v-b\}}^v P_{b-v+2u}\hat{\Gamma}(P_vx,P_vx)dudv\\
&\leq\; 2\int_0^\infty\int_{\max\{0,v-b\}}^v P_{b-\frac{v}{3}+2u}\hat{\Gamma}(P_{\frac{v}{3}}x,P_{\frac{v}{3}}x)dudv\\
&=\; 2\int_0^b\int_0^v P_{b-v+2u}\hat{\Gamma}(P_vx,P_vx)dudv+2\int_b^\infty\int_{v-b}^v P_{b-v+2u}\hat{\Gamma}(P_vx,P_vx)dudv \;=\; I+II\ .
\end{aligned}
$$

For $v \geq b$ we have

$$\frac{b+v}{3} \;\leq\; b-\frac{v}{3}+2u \;\leq\; \frac{5}{3}(b+v)\ .$$

Thus monotonicity implies

$$
\begin{aligned}
II \;&\leq\; 2\int_0^\infty\int_{v-b}^v P_{b-\frac{v}{3}+2u}\hat{\Gamma}(P_{\frac{v}{3}}x,P_{\frac{v}{3}}x)dudv \;\leq\; 10b\int_b^\infty P_{\frac{b+v}{3}}\hat{\Gamma}(P_{\frac{v}{3}}x,P_{\frac{v}{3}}x)dv\\
&= 90\int_{\frac{b}{3}}^\infty P_{\frac{b}{3}+s}\hat{\Gamma}(P_sx,P_sx)\min(s,\frac{b}{3})\ ds\ .
\end{aligned}
$$

In the range $v\leq b$ and $0\leq u\leq v$ we also have

$$\frac{b+v}{3} \;\leq\; b+2u-\frac{v}{3} \;\leq\; \frac{5}{3}(b+v)\ .$$

Again by monotonicity and with $s=\frac{v}{3}$ we obtain

$$I\leq 10\int_0^b P_{\frac{b+v}{3}}\hat{\Gamma}(P_{\frac{v}{3}}x,P_{\frac{v}{3}}x)\ vdv \;=\; 90\int_0^{\frac{b}{3}} P_{\frac{b}{3}+s}\hat{\Gamma}(P_sx,P_sx)\ sds\ .$$

This yields

$$P_b|x|^2-|P_bx|^2\leq 180\int_0^\infty P_{\frac{b}{3}+s}\hat{\Gamma}(P_sx,P_sx)\ \min(\frac{b}{3},s)ds\ . \qquad \blacksquare$$

We will now study different BMO norms motivated by the expressions above, namely

$$
\begin{aligned}
\|x\|_{BMO_c(\hat{\Gamma})} &= \sup_b \|P_b\int_0^b\hat{\Gamma}(P_sx,P_sx)sds\|_\infty^{\frac{1}{2}}\ ,\\
\|x\|_{BMO_c^*(T)} &= \sup_t \|T_t|x-T_tx|^2\|^{1/2}\ .
\end{aligned}
$$

The second norm has been introduced in [Mei08], motivated by the expression

$$\|f\|_{BMO_1} \;=\; \sup_z P_z(|f-f(z)|)\ .$$

We also noticed that it was studied in the commutative case in [DY05b]. With respect to $\|\ \|_{BMO_1}$ it is easy to show that the conjugation operator is bounded from L_∞ to BMO. Here $f(z)$ gives the value of the harmonic extension in the interior of the circle (see [Gar07]). This

means in $f - f(z)$, $f(z)$ is considered as a constant function. In some sense $x - P_t x$ is similar, but clearly $P_t x$ still is a function, even when $P_t x$ is the Poisson integral of f.

Lemma 4.4. *Let (T_t) be a semigroup satisfying $\Gamma^2 \geq 0$. Then*

$$\frac{1}{4}\,||x||^2_{BMO_c(\hat\Gamma)} \;\leq\; ||\sup_t \int_0^\infty P_{s+t}\hat\Gamma(P_s x, P_s x)\min(s,t)ds||^2_\infty \;\leq\; 32\,||x||^2_{BMO_c(\hat\Gamma)}.$$

Proof. For the first estimate we note that due to $\Gamma^2 \geq 0$ we have

$$\|\int_0^t P_t\hat\Gamma(P_v x, P_v x)vdv\|_\infty \;\leq\; \|\int_0^t P_{\frac{v}{2}+t}\hat\Gamma(P_{\frac{v}{2}}x, P_{\frac{v}{2}}x)vdv\|_\infty \;=\; 4\|\int_0^{\frac{t}{2}} P_{s+t}\hat\Gamma(P_s x, P_s x)sds\|_\infty$$
$$\leq\; 4\|\int_0^\infty P_{s+t}\hat\Gamma(P_s x, P_s x)\min(s,t)ds\|_\infty.$$

For the other argument we use a dyadic decomposition. Indeed, according to Proposition 1.3, we have

$$\frac{2^n t P_{s+t}}{s+t}\hat\Gamma(P_s x, P_s x) \leq P_{2^n t}\hat\Gamma(P_s x, P_s x)$$

for $s \geq 2^n t$. This implies

$$\begin{aligned}
\frac{1}{2}\int_0^\infty P_{s+t}\hat\Gamma(P_s x, P_s x)\min(s,t)ds \;&\leq\; \int_0^\infty P_{s+t}\hat\Gamma(P_s x, P_s x)\frac{st}{s+t}ds\\
&= \int_0^{2t} P_t\hat\Gamma(P_s x, P_s x)\frac{st}{s+t}ds + \sum_{n=1}^\infty \frac{1}{2^n}\int_{2^n t}^{2^{n+1}t}\frac{2^n t P_{s+t}}{s+t}\hat\Gamma(P_s x, P_s x)sds\\
&\leq \int_0^{2t} P_t\hat\Gamma(P_s x, P_s x)sds + \sum_{n=1}^\infty \frac{1}{2^n}\int_{2^n t}^{2^{n+1}t} P_{2^n t}\hat\Gamma(P_s x, P_s x)sds\\
&\leq \int_0^{2t} P_t\hat\Gamma(P_s x, P_s x)sds + \sum_{n=1}^\infty \frac{1}{2^n}\int_0^{2^{n+1}t} P_{2^n t}\hat\Gamma(P_s x, P_s x)sds.
\end{aligned}$$

However, we can replace $2t$ by t using $\Gamma^2 \geq 0$ and Lemma 1.3:

$$\int_0^{2t} P_t\hat\Gamma(P_s x, P_s x)sds \;\leq\; \int_0^{2t} P_{t+\frac{s}{2}}\hat\Gamma(P_{\frac{s}{2}}x, P_{\frac{s}{2}}x)sds \;=\; 4\int_0^t P_{t+v}\hat\Gamma(P_v x, P_v x)vdv$$
$$\leq 8\int_0^t P_t\hat\Gamma(P_v x, P_v x)vdv\;.$$

Applying this argument for every $2^{n+1}t$, we deduce the assertion. ■

The next observation is true for arbitrary semigroups (T_t).

Proposition 4.5. *Let (T_t) be a semigroup of completely positive maps. Then*

i) $\|T_s x\|_{BMO_c(T)} \;\leq\; \|x\|_{BMO_c(T)}$ *for all $s > 0$ and $x \in N$;*

ii) $\|x\|_{BMO_c^*(T)} \;\leq\; 2\,\|x\|_{BMO_c(T)} + \sup_t \|T_t x - T_{2t}x\|_\infty^{1/2}$ *for all $x \in N$.*

Proof. Let us start with i) and the pointwise estimate

$$0 \;\leq\; T_t|T_s x|^2 - |T_{t+s}x|^2 \;\leq\; T_{t+s}|x|^2 - |T_{t+s}x|^2\;.$$

By definition of the $BMO_c(T)$ norm this implies

$$\|T_s x\|_{BMO_c(T)} = \sup_t \|T_t|T_s x|^2 - |T_{t+s}x|^2\|_\infty^{\frac{1}{2}} \;\leq\; \sup_t \|T_{t+s}|x|^2 - |T_{t+s}x|^2\|_\infty^{\frac{1}{2}} \;\leq\; \|x\|_{BMO_c(T)}\;.$$

For the proof of ii), we fix $t>0$ and use the triangle inequality (see Lemma 4.1):

$$\begin{aligned}\|T_t|x-T_tx|^2\|_\infty &\le \|T_t|x-T_tx|^2-|T_t(x-T_tx)|^2\|_\infty+\|\,|T_t(x-T_tx)|^2\|_\infty\\ &\le \|x-T_tx\|^2_{BMO_c(T)}+\||T_t(x-T_tx)|^2\|_\infty\\ &\le 2\|x\|^2_{BMO_c(T)}+2\|T_tx\|^2_{BMO_c(T)}+\|T_t(x-T_tx)\|^2_\infty\end{aligned}$$

We apply (i) and obtain

$$\|T_t|x-T_tx|^2\|_\infty \le 4\|x\|^2_{BMO_c(T)}+\|T_t(x-T_tx)\|^2_\infty .$$

Taking supremum over t yields the assertion. ■

Our next goal is to show that the $BMO_c(P)$-norm is in fact larger than the $BMO_c^*(P)$-norm.

Lemma 4.6. *Let $a>1$. Then*

$$\sup_t \|P_tx-P_{at}x\| \ \le\ \sqrt{2}(1+\log_{\frac{3}{2}}a)\|x\|_{BMO_c(\hat{\Gamma}_A)} .$$

Proof. For t fixed, we have the

$$\begin{aligned}|P_{3t}x-P_{2t}x|^2 \ &\le\ P_{\frac{3t}{2}}(|P_{\frac{3t}{2}}x-P_{\frac{t}{2}}x|^2)\ =\ P_{\frac{3t}{2}}(|\int_{\frac{t}{2}}^{\frac{3t}{2}}P_s'xds|^2)\\ &\le\ P_{\frac{3t}{2}}(t\int_{\frac{t}{2}}^{\frac{3t}{2}}|P_s'x|^2ds)\ \le\ 2P_{\frac{3t}{2}}(\int_{\frac{t}{2}}^{\frac{3t}{2}}|P_s'x|^2sds)\ \le\ 2P_{\frac{3t}{2}}(\int_0^{\frac{3t}{2}}|P_s'x|^2sds).\end{aligned}$$

This implies in particular that

$$\sup_t \|P_tx-P_{\frac{3t}{2}}x\|_\infty \ \le\ \sqrt{2}\|x\|_{BMO_c(\hat{\Gamma})} .$$

For $1<a\le\frac{3}{2}$, choose $b\ge 0$ such that $\frac{a-b}{1-b}=\frac{3}{2}$. Then we obtain

$$\begin{aligned}\|\,|P_tx-P_{at}x|^2\|_\infty &\le \|P_{bt}|P_{(1-b)t}(x)-P_{\frac{3}{2}(1-b)t}(x)|^2\|_\infty\\ &\le \|\,|P_{(1-b)t}(x)-P_{\frac{3}{2}(1-b)t}(x)|^2\|_\infty\ \le\ 2\,\|x\|^2_{BMO_c(\hat{\Gamma})}\ .\end{aligned}\tag{4.2}$$

We deduce

$$\|P_t(x)-P_{at}(x)\|_\infty\ \le\ \sqrt{2}\,\|x\|_{BMO_c(\hat{\Gamma}_A)}\tag{4.3}$$

for any $1<a\le\frac{3}{2}$. Consider now $a>\frac{3}{2}$. Let n be the integer part of $\log_{\frac{3}{2}}a$. We may use a telescopic sum

$$P_tx-P_{at}x\ =\ (P_tx-P_{\frac{3t}{2}}x)+(P_{\frac{3t}{2}}x-P_{\frac{3}{2}\frac{3t}{2}}x)+\cdots(P_{(\frac{3}{2})^nt}x-P_{at}x)\ .$$

We apply (4.3) for every summand. Then the triangle inequality implies the assertion. ■

The careful reader will have observed that we need one extra estimate to complete the cycle.

Proposition 4.7. *Let (T_t) be a semigroup satisfying $\Gamma^2\ \ge\ 0$. Then*

$$\|x\|_{BMO_c(\hat{\Gamma})}\ \le\ \frac{18}{3-\sqrt{7}}\,\|x\|_{BMO_c^*(P)}\ .$$

Proof. We fix x,t and split $x=(x-P_{4t}x)+P_{4t}x$. Then we have

$$\begin{aligned}\|\int_0^\infty P_{s+t}\hat{\Gamma}(P_sx,P_sx)\frac{st}{s+t}ds\|_\infty^{\frac{1}{2}}\ &\le\ \|\int_0^\infty P_{s+t}\hat{\Gamma}(P_s(x-P_{4t}x),P_s(x-P_{4t}x))\frac{st}{s+t}ds\|_\infty^{\frac{1}{2}}\\ &\quad+\|\int_0^\infty P_{s+t}\hat{\Gamma}(P_{s+4t}x,P_{s+4t}x)\frac{st}{s+t}ds\|_\infty^{\frac{1}{2}}\end{aligned}$$

For the first term we may apply Proposition 4.3i) for $x' = x - P_{4t}x$ and obtain with $\frac{st}{s+t} \leq \min(s,t)$ that

$$\| \int_0^\infty P_{s+t}\hat{\Gamma}(P_s(x - P_{4t}x), P_s(x - P_{4t}x)) \min(s,t) ds\|_\infty^{\frac{1}{2}} \;\leq\; 2\, \|P_t|x - P_{4t}x|^2\|^{1/2} \,.$$

The last term can be estimated by the BMO_c^*-norm using the triangle inequality from Lemma 4.1 as follows

$$\|P_t|x - P_{4t}x|^2\|^{1/2} \;\leq\; \|P_t|x - P_tx|^2\|^{1/2} + \|P_t|P_tx - P_{2t}x|^2\|^{1/2} + \|P_t|P_{2t}x - P_{4t}x|^2\|^{1/2}$$

$$\leq \|P_t|x - P_tx|^2\|^{1/2} + \|P_{2t}|x - P_tx|^2\|^{1/2} + \|P_{3t}|x - P_tx|^2\|^{1/2} \;\leq\; 3\|x\|_{BMO_c^*(P)} \tag{4.4}$$

For the tail estimate we apply again the Γ^2 condition:

$$\| \int_0^\infty P_{s+t}\hat{\Gamma}(P_{s+4t}x, P_{s+4t}x)\frac{st}{s+t} ds\|_\infty^{\frac{1}{2}} \;\leq\; \| \int_0^\infty P_{s+t+3t}\hat{\Gamma}(P_{s+t}x, P_{s+t}x)\frac{st}{s+t} ds\|_\infty^{\frac{1}{2}}$$

$$= \| \int_t^\infty P_{s+3t}\hat{\Gamma}(P_sx, P_sx)\frac{(s-t)t}{s} ds\|_\infty^{\frac{1}{2}} \,.$$

For $s \geq t$ we consider the function $f(s) = \frac{(s-t)(s+3t)}{3s^2} = \frac{1}{3} + \frac{t}{s}[\frac{2}{3} - \frac{t}{s}]$. Note that $f(t) = 0$ and for $t \leq s \leq \frac{3}{2}t$ we have $f(s) \leq \frac{1}{3}$. For $s \geq \frac{3}{2}t$ we $f(s) \leq \frac{1}{3} + \frac{4}{9} = \frac{7}{9}$. Thus in any case

$$\frac{(s-t)t}{s} \leq \frac{7}{9}\frac{3st}{s+3t}.$$

Taking supremum over t, we obtain

$$\sup_t \| \int_0^\infty \int_0^\infty P_{s+t}\hat{\Gamma}(P_sx, P_sx)\frac{st}{s+t} ds\|_\infty^{\frac{1}{2}}$$

$$\leq\; 2\sup_t \|P_t|x - P_{4t}x|^2\|^{1/2} + \frac{\sqrt{7}}{3}\sup_t \| \int_0^\infty \int_0^\infty P_{s+3t}\hat{\Gamma}(P_sx, P_sx)\frac{s3t}{s+3t} ds\|_\infty^{\frac{1}{2}} \,.$$

This implies

$$(1 - \frac{\sqrt{7}}{3})\sup_t \| \int_0^\infty \int_0^\infty P_{s+t}\hat{\Gamma}(P_sx, P_sx)\frac{st}{s+t} ds\|_\infty^{\frac{1}{2}} \;\leq\; 2\, \sup_t \|P_t|x - P_{4t}x|^2\|^{\frac{1}{2}}$$

$$\leq\; 6\, \|x\|_{BMO_c^*(P)} \,.$$

We may replace $\frac{st}{s+t}$ by $\min(s,t)$ with an additional factor 2. Hence the assertion follows from Lemma 4.4. ■

Theorem 4.8. *Let (T_t) be semigroup of completely positive maps satisfying $\Gamma^2 \geq 0$. Then the norms $\| \, \|_{BMO_c(P)}$, $\| \, \|_{BMO_c^*(P)}$ and $\| \, \|_{BMO_c(\hat{\Gamma})}$ are all equivalent on $\mathcal{A}$.*

Proof. According to Proposition 4.3 we know that

$$\sup_t \| \int_0^\infty P_{s+t}\hat{\Gamma}(P_sx, P_sx) \min(s,t) ds\|_\infty^{\frac{1}{2}} \sim_{180} \|x\|_{BMO_c(P)} \,.$$

Then Lemma 4.4 implies that $\| \, \|_{BMO_c(P)}$ and $\| \, \|_{BMO_c(\hat{\Gamma})}$ are equivalent. Proposition (4.7) provides the upper estimate of $\| \, \|_{BMO_c(\hat{\Gamma})}$ against $\| \, \|_{BMO_c^*(P)}$. Conversely, we deduce from Proposition 4.5, Lemma 4.6, Lemma 4.4 and Proposition 4.3 i) that

$$\|x\|_{BMO_c^*(P)} \;\leq\; 2\|x\|_{BMO_c(P)} + \sup_t \|P_tx - P_{2t}x\|$$

$$\leq 2\|x\|_{BMO_c(P)} + \sqrt{2}(1 + \log_{\frac{3}{2}} 2)\|x\|_{BMO_c(\hat{\Gamma})}$$

$$\leq 2\|x\|_{BMO_c(P)} + 2\sqrt{2}\, 2\sqrt{6}\, \|x\|_{BMO_c(P)} \;=\; (2 + 8\sqrt{3})\|x\|_{BMO_c(P)} \,.$$

Thus all the norms are equivalent on $\mathcal{A}$. ■

We conclude this section with two results on interpolation which show that the BMO spaces are indeed a good endpoints. Let us define

$$\|x\|_{BMO(T)} \;=\; \max\{\|x\|_{BMO_c(T)}, \|x^*\|_{BMO_c(T)}\}$$

and the space $BMO(T)$ as the completion of N with respect to that norm.

Theorem 4.9. *Let (T_t) be a semigroup of completely positive maps with a Markov dilation and $\Gamma^2 \;\geq\; 0$. Then*

i) $[BMO(T), L_p(N)]_{\frac{1}{q}} \;=\; L_{pq}(N)$;

ii) $[BMO(P), L_p(N)]_{\frac{1}{q}} \;=\; L_{pq}(N)$.

holds for $1 \leq p < \infty$, $1 < q < \infty$.

Proof. For both proofs we note that the trivial inclusion $N \subset BMO$ implies

$$L_{pq}(N) \subset [BMO(N), L_p(N)]_{\frac{1}{q}} \;.$$

For the converse in i) we consider the norm

$$\|\pi_0(x)\|_{L_p^cmo(N)} \;=\; \|\sup_t \pi_t(T_t|x|^2 - |T_tx|^2)\|_{L_{p/2}}^{1/2} \;. \tag{4.5}$$

Since $\Gamma^2 \;\geq\; 0$, we know that the reversed martingale $m_t(x) = \pi_t(T_t(x))$ has continuous path and therefore

$$\|\pi_0(x)\|_{L_p^cMO} \;=\; \lim_{|\sigma|,\mathcal{U}} \|\sup_j E_{t_j]}(|\pi_0(x) - m_{t_j}(x)|^2)\|_{p/2} \;=\; \|x\|_{L_p^cmo}$$

holds for every ultrafilter on the set of partitions. For a fixed partition and a reversed martingale (y_s) we introduce

$$\|y\|_{L_p^cMO(\sigma)} \;=\; \max\{\|\sup_j E_{t_j]}(|y_0 - y_{t_j}|^2)\|_{p/2}^{1/2}, \|\sup_j E_{t_j]}(|y_0^* - y_{t_j}^*|^2)\|_{p/2}^{1/2}\} \;.$$

It was shown in [JM07] that

$$[L_pMO(\sigma), L_q(M)]_\theta \subset L_s(M) \quad \text{with} \quad \frac{1}{s} = \frac{1-\theta}{p} + \frac{\theta}{q}$$

and the constant $c(s)$ is uniformly on compact intervals of $(1,\infty)$ and independent of σ. Thus for the norm

$$\|y\|_{L_pMO} \;=\; \lim_{\sigma,\mathcal{U}} \|x\|_{L_pMO(\sigma)}$$

we still have

$$[L_pMO, L_q(M)]_\theta \subset L_s(M) \quad \text{with} \quad \frac{1}{s} = \frac{1-\theta}{p} + \frac{\theta}{q} \;.$$

However, on $\pi_0(A)$ we know that the L_pMO and the L_pmo norm coincide and hence

$$[\overline{\pi_0(\mathcal{A})}^{\|\;\|_{L_pmo}}, \overline{\pi_0(\mathcal{A})}^{\|\;\|_q}]_\theta \subset L_s(M) \cap \overline{\pi_0(\mathcal{A})}^{\|\;\|_s} \;.$$

Let us briefly indicate that $\pi_0(\mathcal{A})$ is dense in $\pi_0(N)$ viewed as a subspace of L_pMO. Indeed, since the continuous martingales are closed with respect to the L_p norm, and the $L_p(M)$ majorizes the L_p^cMO norm up to a constant $c(p)$, the density follows from the density if $\mathcal{A}$ in $L_p(N)$. Finally, the embedding π_0 of $L_s(N)$ in $L_s(\hat{M})$ is isometric and therefore we have

$$[L_pmo(N), L_q(N)]_\theta \subset L_s(N)$$

with a constant $c(s)$ bounded on compact intervals. Sending $p \to \infty$ and θ to $\frac{1}{v}$, we deduce

$$[BMO(T), L_q(N)]_{\frac{1}{v}} \subset L_{qv}(N) \ .$$

For P we can either work with the Markov dilation ρ_a and perform a similar argument, or we can observe that a Markov dilation for (T_t) produces a Markov dilation for (P_t). ■

In some applications it is worth while to work with the column space $BMO_c(T)$ as completion of N with respect to the $BMO_c(T)$ norm.

Theorem 4.10. *Let (T_t) be a semigroup of completely positive maps with a Markov dilation and $\Gamma^2 \ \geq\ 0$. Then*

i) $[BMO_c(P), L_2(N)]_{\frac{2}{p}} \subset H_p^c(P)$;

ii) $[BMO_c(T), L_2(N)]_{\frac{2}{p}} \subset H_p^c(T)$.

Proof. For the first assertion, we just recall that as in the proof of Theorem 4.9 we have

$$[L_p^c MO(M), L_2(M)]_\theta \subset H_s^c(M) \quad , \quad \frac{1}{s} \ = \ \frac{1-\theta}{p} + \frac{\theta}{2} \ .$$

The same remark on the uniformity of constants applies. Note also that

$$\|\pi_0(x)\|_{L_q^c MO(M)} \ \leq \ \|\pi_0(x)\|_{BMO_c(M)} \ \leq \ \|x\|_{BMO_c(P)} \ .$$

Sending $p \to \infty$ and applying Theorem 2.9ii) we obtain assertion ii). Assertion i) can be derived from ii) or adapting the argument for ρ_a instead of π_0. ■

5. Abstract semigroup theory

In the theory of semigroups certain tools from classical Hardy-Littlewood theory are still available. In this section we will recall the so-called Hardy-Littlewood-Sobolev theory which is beautifully presented in [VSCC92]. Almost all (but not all) the methods from the commutative theory apply in our setting. In particular, we will prove the von Neumann algebra version of [VSCC92, Theorem II.5.2]. We refer to [VSCC92] for history and credits. We are interested in the space

$$L_p^0(N) \ = \ (\mathrm{I} - \mathrm{Pr})L_p(N)$$

of mean 0 elements. Recall that $\mathrm{Pr} = \lim_{t\to\infty} T_t$ is the orthogonal projection onto the kernel of A and hence $(\mathrm{I} - \mathrm{Pr})$ is a complete bounded (with cb-norm ≤ 2) on all $L_p(N)$.

Theorem 5.1. *Let (T_t) be a semigroup of completely positive selfadjoint contractions on a von Neumann algebra N with negative generator A and $n > 2$. Let $L_p^0(N)$ be the space of mean 0 elements. The following are equivalent*

i) $\|x\|_{2n/(n-2)}^2 \ \leq\ C_1\,(x, Ax)$ *for all mean 0 elements x,*

ii) $\|x\|_2^{2+4/n} \ \leq\ C_2\,(x, Ax)\,\|x\|_1^{4/n}$ *for all mean 0 elements x,*

iii) $\|T_t : L_1^0(N) \to L_\infty(N)\| \ \leq\ C_3 t^{-n/2}$.

An important tool is the family of conditions

$$(\mathrm{R}_n^{pq}) \qquad \|T_t : L_p^0(N) \to L_q^0(N)\| \ \leq\ Ct^{-\frac{n}{2}(1/p-1/q)} \quad , \quad 1 \leq p \leq q \leq \infty$$

The proof of the following result is verbatim the same as in the commutative case.

Lemma 5.2. *Let T_t be a selfadjoint family of operators, uniformly bounded on $L_p(N)$. Then (R_n^{pq}) holds for one pair $1 \leq p < q \leq \infty$ if and only if for all $1 \leq p \leq q \leq \infty$.*

Sketch of proof. Let $p_1 \leq p < q$ and $\frac{1}{p} = \frac{1-\theta}{p} + \frac{\theta}{q}$. Assume (R_n^{pq}). Then we deduce from interpolation that

$$\|T_t(x)\|_p \;\leq\; Ct^{-\frac{n}{2}(1/p-1/q)}\|x\|_p \;\leq\; Ct^{-\frac{n}{2}(1/p-1/q)}\|x\|_{p_1}^{1-\theta}\|x\|_q^{\theta} \,.$$

Thus $(R_n^{p_1q})$ holds with constant $C^{1/1-\theta}$. In particular, (R_n^{1q}) holds. By duality we find $(R_n^{q'\infty})$. Applying the argument again we get $(R_n^{1\infty})$. Now, we show that $(R_n^{1,\infty})$ implies (R_n^{pq}). Indeed, by complementation and interpolation we have

$$\|T_t : L_p^0(N) \to L_\infty\| \;\leq\; 2\|T_t : L_\infty^0(N) \to L_\infty\|^{1-1/p}\|T_t : L_1^0(N) \to L_\infty(N)\|^{1/p} \,.$$

This yields $(R_n^{p\infty})$. The same interpolation argument implies (R_n^{pq}). ■

In the following we will simply refer to the condition

$$(\mathrm{R}_n) \qquad \|T_t : L_1^0(N) \to L_\infty(N)\| \;\leq\; Ct^{-\frac{n}{2}}$$

Our next result requires a little bit more interpolation theory. We recall that two Banach spaces $A_0, A_1 \subset V$ are injectively embedded in a common topological vector space such that $A_0 \cap A_1$ is dense in A_0 and A_1. The unit ball of the space $[A_0, A_1]_{\theta,1}$ is the convex hull of element x in the intersection satisfying

$$\|x\|_{A_0}^{1-\theta}\|x\|_{A_1}^{\theta} \leq 1 \,.$$

This implies that a linear operator $T : [A_0, A_1]_{\theta,1} \to X$ with values in a Banach space is continuous if

$$(\theta, 1) \qquad \|T(x)\| \;\leq\; C\|x\|_0^{1-\theta}\|x\|_1^{\theta} \quad , \quad x \in X \,.$$

The corresponding "dual" observation holds for the interpolation space $[A_0, A_1]_{\theta,\infty}$. We recall that the norm of x in $[A_0, A_1]_{\theta,\infty}$ is less than C if for every $t > 0$ we can decompose $x = x_0 + x_1$ such that

$$(\theta, \infty) \qquad \|x_0\|_0 + t\|x_1\|_1 \;\leq\; Ct^{\theta} \,.$$

We will apply this for the scale of noncommutative Lorentz spaces

$$L_{r,s}(N) \;=\; [L_{p,s_1}(N), L_{q,s_2}(N)]_{\theta,s} \quad , \quad \frac{1}{r} \;=\; \frac{1-\theta}{p} + \frac{\theta}{q} \tag{5.5}$$

which holds for all $1 \leq s_1, s_2 \leq \infty$ and $0 < \theta < 1$. We refer to [BL96] for general information on interpolation theory and to [PX03] for the translation to the noncommutative setting. Since the space $L_p^0(N)$ is completely complemented and $L_{p,p}(N) = L_p(N)$, we may define $L_{r,s}^0(N) = [L_p^0(N), L_q^0(N)]_{\theta,s}$ and then (5.5) remains true for the spaces $L_{r,s}^0(N)$. The next argument is adapted from [Var85]. The conclusion is slightly weaker than in the commutative situation. The key ingredient is the resolvent formula

$$A^{-z} \;=\; \Gamma(z)^{-1}\int_0^\infty T_t\, t^{z-1}dt \quad \text{ for } Re(z) > 0. \tag{5.6}$$

Lemma 5.3. *Let (T_t) be a semigroup of normal selfadjoint contractions such that (R_n) holds. Let $z \in \mathbb{C}$ and $\alpha = Re(z)$.*

i) *Let $1 \leq p < s < q \leq \infty$ and $z \in \mathbb{C}$ with $\alpha = \frac{n}{2}(\frac{1}{s} - \frac{1}{q})$. Then*

$$\|A^{-z} : L_{s,1}^0(N) \to L_q(N)\| \;\leq\; C(\alpha, n) \,.$$

ii) *Let $1 \leq p < r < \infty$ such that $\alpha = \frac{n}{2}(\frac{1}{p} - \frac{1}{r})$. Then*

$$\|A^{-z} : L_p^0(N) \to L_{r,\infty}(N)\| \;\leq\; C(\alpha, n) \,.$$

Proof. Ad i): We define $\alpha = Re(z)$, $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ and $\theta = \frac{2r\alpha}{n}$. Let x be an element in $L_q^0(N)$ and $b > 0$. In combination with (R_n), we deduce from (5.6) that

$$|\Gamma(z)|\|A^{-z}(x)\|_q \;=\; \|\int_0^\infty T_t(x)t^{z-1}dt\|_q \;\le\; \int_0^b \|T_t(x)\|_q\, t^{\alpha-1}dt + \int_b^\infty Ct^{-n/2r}\|x\|_p\, t^{\alpha-1}dt$$
$$\le \alpha^{-1}b^\alpha\, \|x\|_q + C(\frac{n}{2r}-\alpha)^{-1}b^{\alpha-\frac{n}{2r}}\, \|x\|_p\, .$$

We choose $b^{\frac{n}{2r}} \;=\; \frac{\|x\|_p}{\|x\|_q}$. (For $\|x\|_q = 0$ there is nothing to show.) This yields

$$\|A^{-z}(x)\|_q \;\le\; |\Gamma(z)|^{-1}K'\frac{n}{\alpha(n-2r\alpha)}\|x\|_q^{1-\frac{2r\alpha}{n}}\|x\|_p^{\frac{2r\alpha}{n}}\, .$$

The assertion follows from equation $(\theta, 1)$ and $L_{s,1}^0(N) \;=\; [L_q^0(N), L_p^0(N)]_{\theta,1}$. Note also that $1/s = (1-\theta)/q + \theta/p = 1/q + \theta/r = 1/q + 2\alpha/n$. For the proof of ii) we define $\theta = 1 - \frac{2p\alpha}{n}$. Assume that $x \in L_p^0(N)$ and decompose $\Gamma(z)A^{-z}x = x_0 + x_1$ where

$$x_0 \;=\; \int_b^\infty T_t(x)t^{z-1}dt \quad , \quad x_1 \;=\; \int_0^b T_t(x)t^{z-1}dt\, .$$

As above we deduce from $R_n^{p,\infty}$ and the assumption $2p\alpha < n$ that

$$\|x_0\|_\infty \;\le\; \frac{2Cp}{n-2p\alpha}b^{\alpha-\frac{n}{2p}}\, \|x\|_p\, .$$

On the other hand, we have $\|x_1\|_p \le \frac{b^\alpha}{\alpha}\|x\|_p$. For fixed $t > 0$ we choose b such that $b^{-n/2p} = t$. Then

$$\|x_0\|_\infty + t\|x_1\|_p \;\le\; K'\|x\|_p\, b^{\alpha-\frac{n}{2p}} \;=\; K'\|x\|_p\, t^{1-\frac{2p\alpha}{n}}\, .$$

Thus we have verified condition (θ, ∞) and the assertion follows from the real interpolation method $L_{r,\infty}(N) = [L_\infty(N), L_p(N)]_{\theta,\infty}$. ∎

As an immediate application of the Marcinkiewicz interpolation theorem (in the form of (5.5)), we can remove the Lorentz spaces from the conclusion.

Corollary 5.4. *Let (T_t) be a semigroup of normal selfadjoint contractions such that (R_n) holds. In the type III case we assume in addition that T_t commutes with the modular group of a normal faithful state. Let $z \in \mathbb{C}$ and $\alpha = Re(z)$. Then*

$$\|A^{-z} : L_p^0(N) \to L_q^0(N)\| \;\le\; C(\alpha)$$

holds for all $1 < p < q < \infty$ such that $\alpha \;=\; \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$.

Proof of Theorem 5.1. For the proof of the implication iii) $\Rightarrow$ i) we choose $z = \frac{1}{2}$ and $p = 2$. Note that $(x, Ax) = \|A^{1/2}x\|_2^2$. Since (R_n) is satisfied, we obtain $1 = n(1/2 - 1/q)$, i.e. $q = 2n/(n-2)$. The implication i) $\Rightarrow$ ii) follows from

$$\|x\|_2 \;\le\; \|x\|_{\frac{2n}{n-2}}^{\frac{n}{n+2}}\|x\|_1^{\frac{2}{n+2}}\, .$$

The implication ii) $\Rightarrow$ iii) follows verbatim as in [VSCC92, Theorem III.3.2]. One first shows $(R_n^{1,2})$ by differentiation for a selfadjoint mean 0 element x using $\frac{d}{dt}\|T_tx\|_2^2 = -2Re(AT_tx, T_tx)$. Remark 5.2 implies the assertion. ∎

For our applications we need compactness results of the operator $A^{-\alpha}$ on $L_p^0(N)$. Our aim is to derive them from compactness on $L_2^0(N)$. Let us consider the following conditions

gap$_c$) The spectrum of $(1 - \mathrm{Pr})A$ on $L_2(N)$ is contained in $[c, \infty)$,
com) A^{-1} is compact on $L_2^0(N)$.

Proposition 5.5. *Let A be a generator which satisfies gap_c. Then*

i) *Let $Re(z) > 0$. Then A^{-z} is (completely) bounded on $L_p^0(N)$ for $1 < p < \infty$,*

ii) *$\|T_t : L_p^0(N) \to L_p^0(N)\|_{cb} \le 2e^{-\frac{2tc}{p}}$ for all $2 \le p < \infty$.*

Proof. First we note that gap$_c$) means $A \ge c$ on the Hilbert space $L_2^0(N)$ and hence

$$\|e^{-tA} : L_2^0(N) \to L_2^0(N)\| \le e^{-tc} .$$

Let $2 \le p < \infty$. Since $L_p^0(N)$ forms an interpolation scale, we deduce

$$\|T_t : L_p^0(N) \to L_p^0(N)\| \le 2\|T_t : L_2^0(N) \to L_2^0(N)\|^{\frac{2}{p}} \|T_t : N \to N\|^{1-2/p} \le 2e^{-2tc/p} .$$

This shows ii) and the same argument for $T_t \otimes id$ provides the cb-estimate. For the proof of i) we assume $2 < p < \infty$. Then (5.6) implies with $\alpha = Re(z)$ that

$$\|A^{-z} : L_p^0(N) \to L_p^0(N)\|_{cb} \le 2|\Gamma(z)|^{-1} \int_0^\infty e^{-2ct/p} s^{\alpha-1} ds < \infty .$$

Since A is selfadjoint the same estimate holds on $L_{p'}^0(N)$. ■

The next Lemma allows to interpolate compactness (see [Pie80]).

Lemma 5.6. *Let (A_0, A_1) be an interpolation couple as above, $T : X \to A_0 \cap A_1$ a linear map such that $T : A \to A_0$ is bounded and $T : X \to A_1$ is compact. Then $T : X \to A_\theta$ is compact.*

Proof. Let us recall that $T : X \to Y$ is compact if and only if the entropy numbers

$$e_k(T) = \inf\{\varepsilon : T(B_X) \subset \bigcup_{j=0}^{2^{k-1}} y_j + \varepsilon B_Y\}$$

satisfy $\lim_k e_k(T) = 0$. Here B_X, B_Y, is the unit ball of X, Y, respectively. The infimum is taken over arbitrary points in y. We recall from [Pie80] that

$$e_{k+j-1}(T : X \to A_\theta) \le 2e_k(T : X \to A_0)^{1-\theta} e_j(T : X \to A_1)^\theta .$$

In particular, $e_k(T : X \to A_\theta) \le 2\|T : X \to A_0\|^{1-\theta} e_k(T : X \to A_1)^\theta$ still converges to 0. ■

Theorem 5.7. *Let (T_t) be a semigroup of selfadjoint, positive contractions on a finite von Neumann algebra satisfying*

$$\|T_t : L_1^0(N) \to L_\infty(N)\| \le Ct^{-n/2}$$

and such that A^{-1} is compact on $L_2^0(N)$. Then $A^{-z} : L_p^0(N) \to L_q^0(N)$ is compact for all $1 \le p < q \le \infty$ such that $\frac{2Re(z)}{n} > \frac{1}{p} - \frac{1}{q}$.

Proof. By assumption A^{-1} is bounded on $L_2^0(N)$ and hence we have a spectral gap. Now, we consider $2 < p < \infty$ and want to show that $A^{-\alpha} : L_p^0(N) \to L_p^0(N)$ is compact for all $\alpha > 0$. According to Proposition 5.5 it suffices to consider $\alpha < n/2p$. Define $1/q = 1/p - 2\alpha/n$. According to Corollary 5.4 we know that $A^{-\alpha} : L_p^0(N) \to L_q^0(N)$ is bounded. Since $A^{-\alpha}$ is compact on $L_2^0(N)$ we also know that $A^{-\alpha} : L_p^0(N) \to L_2^0(N)$ is compact. We may write $L_p^0(N) = [L_q^0(N), L_2^0(N)]_\theta$ where $1/p = (1-\theta)/q + \theta/2$ and $0 < \theta < 1$. Hence Lemma 5.6 implies that $A^{-\alpha} : L_p^0(N) \to L_p^0(N)$ is compact. By duality we conclude that $A^{-\alpha} : L_p^0(N) \to L_p^0(N)$ is compact for all $\alpha > 0$ and $1 < p < \infty$.

Now, we consider $z = \alpha + is$ and assume that $1 < p < q$. By our assumption $2\alpha/n > 1/p - 1/q$. This allows us to find $1 < s < p$, and $\alpha_1 > 0$ such that $2(\alpha - \alpha_1)/n = 1/s - 1/q$. According

to Lemma 5.3i) we know that $A^{\alpha_1 - z} : L^0_{s,1}(N) \to L^0_q(N)$ is bounded. On the other hand $A^{-\alpha_1} : L^0_p(N) \to L^0_p(N)$ is compact and the inclusion $L^0_p(N) \subset L^0_{s,1}(N)$ continuous. Then

$$A^{-z} = A^{\alpha_1 - z} A^{-\alpha_1} : L^0_p(N) \overset{A^{-\alpha_1}\text{compact}}{\longrightarrow} L^0_p(N) \subset L^0_{s,1}(N) \overset{A^{\alpha_1 - z}}{\longrightarrow} L^0_q(N)$$

is the composition of a bounded operator and a compact operator, hence itself compact.

In the case $1 \le p < q < \infty$ we use the same argument and find $q < r < \infty$, a decomposition $A^{-z} = A^{-\alpha_1} A^{\alpha_1 - z}$ such that $A^{\alpha_1 - z} : L^0_p(N) \to L^0_{r,\infty}(N)$ is continuous, $A^{-\alpha_1} : L^0_q(N) \to L^0_q(N)$ is compact, and the inclusion $L^0_{r,\infty}(N) \subset L^0_q(N)$ is continuous. Thus A^{-z} is compact. Finally for $p = 1$ and $q = \infty$ we write $A^{-z} = A^{-z/2} A^{-z/2}$ and make a pit stop at $L^0_2(N)$. ■

6. Applications

6.1. **Quantum metric spaces.** We recall from [Rie98] that a quantum metric space is given by a C^*-algebra C, a *-subalgebra $\mathcal{A}$ and a norm $|||\ |||$ on $\mathcal{A}$ such that

$$d_{|||\ |||}(\phi, \psi) \ = \ \sup\{|\phi(a) - \psi(a)| : a \in \mathcal{A}, |||\, a\, ||| \le 1\}$$

induces the weak* topology on the state space $S(C)$. A norm $|||\ |||$ is a Lipschitz norm if in addition

$$|||\, ab\, ||| \ \le \ |||\, a\, |||\, \|b\| + \|a\|\, |||\, b\, ||| \ . \tag{6.1}$$

In [OR05] a Lipschitz norm generating the weak* topology is called a Lip-norm, i.e. quantum metric space require a Lip-norm instead of Lipschitz norm.

Lemma 6.1. *Let T_t be a unital completely positive semigroup on a von Neumann algebra N. Let $-A$ be the generator and $\mathcal{A}$ be a (non-complete) *-algebra contained in the domain of A. Then*

$$\|a\|_\Gamma \ = \ \max\{\|\Gamma(a,a)\|^{1/2}, \|\Gamma(a^*, a^*)\|^{1/2}\}$$

and $|||\, a\, ||| = \|\Gamma(a,a)\|^{1/2}$ satisfy (6.1).

Proof. We recall from [Pet] that $H_N = \{\sum_i a_i \otimes y_i : \sum_i x_i y_i = 0\}$ equipped wit the N-valued inner product

$$\langle a_1 \otimes x_1, a_2 \otimes x_2 \rangle \ = \ x_1^* \Gamma(a_1, a_2) x_2$$

defines a N-valued Hilbert module. Then $\delta(a) = a \otimes 1 - 1 \otimes a$ is a derivation, i.e.

$$\delta(ab) \ = \ ab \otimes 1 - 1 \otimes ab \ = \ (a \otimes 1)(b \otimes 1 - 1 \otimes b) + (a \otimes 1 - 1 \otimes a)(1 \otimes b) \ = \ (a \otimes 1)\delta(b) + \delta(a)(1 \otimes b) \ .$$

Since $T_t(1) = 1$ we have $A(1) = 0$ and

$$\Gamma(1, a) \ = \ 1A(a) + a^* A(1) - A(1a) \ = \ 0 \ .$$

Hence $\langle \delta(a), \delta(a) \rangle = \Gamma(a,a)$. This implies

$$\|\Gamma(ab, ab)\|^{1/2} \ = \ \|\delta(ab)\| \ \le \ \|(1 \otimes a)\delta(b)\| + \|\delta(a)(1 \otimes b)\| \ \le \ \|a\|\|\delta(b)\| + \|\delta(a)\|\|b\| \ .$$

Recall that $\|\delta(a)\| = \|\Gamma(a,a)\|^{1/2}$. This also shows

$$\|\Gamma((ab)^*, (ab)^*)\|^{1/2} \ = \ \|\Gamma(b^* a^*, b^* a^*)\|^{1/2} \ \le \ \|b^*\|\|\Gamma(a^*, a^*)\|^{1/2} + \|\Gamma(b^*, b^*)\|^{1/2}\|a\| \ .$$

Taking the maximum yields (6.1). ■

We also need the following observation from [OR05, Proposition 1.3]

Lemma 6.2. *Let $|||\ |||$ be a Lipschitz norm and σ be a state. Then $(C, \mathcal{A}, |||\ |||)$ is a quantum metric space iff*

$$\{x \in \mathcal{A} \ : \ |||\, a\, ||| \le 1, \sigma(a) = 0\}$$

is relatively compact in C.

Theorem 6.3. *Let (T_t) be a completely positive semigroup of selfadjoint maps on a finite von Neumann algebra N such that $\mathcal{A} \subset N$ is weakly dense and with a Markov dilation. Assume in addition*

i) $\ker(A) = \mathbb{C}1$ *and* A^{-1} *is compact on* $L_2^0(N) = (I - \mathrm{Pr})L_2(N)$,

ii) $\|T_t : L_2^0(N) \to N\| \;\le\; Ct^{-n/4}$ *for some* $n > 0$.

Then

$$\|x\|_\Gamma \;=\; \max\{\|\Gamma(x,x)^{1/2}\|, \|\Gamma(x^*,x^*)^{1/2}\|\}$$

and $|\!|\!|\, x \,|\!|\!| = \|\Gamma(x,x)\|^{1/2}$ *define a quantum metric spaces for the norm closure* C *of* $\mathcal{A} \subset N$.

Proof. Let us recall that Pr is the projection onto the kernel of the selfadjoint operator A. Thus i) implies in particular that $\ker(A) = \mathbb{C}1$ and that A has a spectral gap

$$c = \|A^{-1} : L_2^0(N) \to L_2^0(N)\| \,.$$

Moreover, $\lim_{t\to\infty} T_t(x) = \tau(x)1$ and hence $L_p^0(N)$ is the closure of elements $x \in N$ such that $\tau(x) = 0$. Let $1 < s < p < \infty$ such that $2 < p$ and $\frac{2\alpha}{n} > \frac{1}{p}$. According to Corollary 5.7 we know that

$$\{x \in L_p^0 : \|A^\alpha x\|_p \le 1\} \subset L_\infty^0(N)$$

is relatively compact in N. Let $\delta > 0$. Then we deduce from [Jun08] and Theorem 3.15 that

$$\begin{aligned}\|A^{\frac12-\delta}x\|_p \le c(\delta)\|A^{\frac12}x\|_{H_p^c(T)} \;&\le\; c(\delta)c(p)\|\Gamma(x,x)^{1/2}\|_p \\ &\le c(\delta)c(p)\|\Gamma(x,x)^{1/2}\|_\infty \;=\; c(\delta)c(p)\|\Gamma(x,x)\|_\infty^{1/2} \,.\end{aligned}$$

Hence we need $\frac12 - \delta > \frac{n}{2p}$ which is satisfied for $p > n$. Lemma 6.2 implies the assertion. ∎

Remark 6.4. *Let M be a compact Riemannian manifold. Then*

$$d(p,q) \;=\; \sup\{|f(x) - f(y)| \;:\; \|\,|\nabla f|\,\|_\infty \le 1\} \,.$$

Moreover, $\Gamma(f,f) = |\nabla f|^2$. The condition ii) corresponds to a Sobolev embedding theorem and Theorem 6.3 provides an appropriate gradient norm in this context.

6.2. **Rapid decay and quantum metric spaces.** Let us recall that a finitely generated discrete group has *rapid decay (RD) of order s* if there exists an $s < \infty$ such that

$$\|x\|_\infty \;\le\; C(s)k^s\|x\|_2$$

holds for all linear combinations $x = \sum_{|g|=k} a_g\lambda(g)$. Here $|\;|$ is the word length function with respect to fixed number of generators. The notion is, however, independent of that choice. We refer to [Jol90] for more information. The following observation is closely related to the work of Rieffel and Ozawa [OR05].

Lemma 6.5. *Let G be a discrete, finitely generated group with word length function $|\;|$ and rapid decay of order s. Let $\psi : G \to \mathbb{R}$ be a conditionally negative function such that*

$$\inf_{l(g)=k} \psi(g) \;\ge\; c_\alpha k^\alpha \tag{6.2}$$

for some $\alpha > 0$. Then the operator $T_t(\lambda(g)) = e^{-t\psi(g)}\lambda(g)$ satisfies

$$\|T_t : L_2^0(N) \to N\| \;\le\; C(s,\alpha)\, t^{-\frac{2s+1}{2\alpha}} \,.$$

Proof. We consider a decomposition $x = \sum_k x_k$ such that $x_k = \sum_{|g|=k} a_g \lambda(g)$ is supported by words of length k. Note that $T_t(x_k)$ is still supported on words of length k and for such g we have $e^{-t\psi(g)} \le e^{-tc_\alpha k^\alpha}$. Hence we get

$$\begin{aligned} \|T_t x\| &\le \sum_k \|T_t x\|_\infty \ \le\ C(s) \sum_k k^s \|T_t x\|_2 \\ &\le C(s) \sum_k k^s e^{-tc_\alpha k^\alpha} \|x_k\|_2 \ \le\ C(s) (\sum_k k^{2s} e^{-2tc_\alpha k^\alpha})^{1/2} (\sum_k \|x_k\|_2^2)^{1/2} . \end{aligned}$$

Now it remains to estimate the sum via some calculus (i.e. $y = 2tc_\alpha x^\alpha$, $dy/y = \alpha dx/x$)

$$\begin{aligned} \sum_{k \ge 1} k^{2s} e^{-2tc_\alpha k^\alpha} &= e^{-2tc_\alpha} + 2^{2s} \int_{2c_\alpha}^\infty x^{2s+1} e^{-t2c_\alpha x^\alpha} \frac{dx}{x} \\ &= e^{-2tc_\alpha} + 2^{2s} \alpha^{-1} (2tc_\alpha)^{-(2s+1)/\alpha} \int_1^\infty y^{\frac{2s+1}{\alpha}} e^{-y} \frac{dy}{y} . \end{aligned}$$

Thus for $0 < t \le 2$ we obtain

$$\|T_t x\|_\infty \ \le\ C(s,\alpha)\, t^{-\frac{2s+1}{2\alpha}} \|x\|_2 .$$

We recall that on $L_2^0(N) = \mathbb{C}1^\perp$ we have a spectral gap $\psi(w) \ \ge\ c(\alpha)|w|^\alpha \ \ge\ c(\alpha)$ for all $w \ne 1$. Hence $\|T_t : L_2^0(N) \to L_2^0(N)\| \le e^{-tc_\alpha}$. Hence for $t \ \ge\ 2$ we have

$$\|T_t x\|_\infty \ =\ \|T_1(T_{t-1}x)\|_\infty \ \le\ C(s,\alpha)\, \|T_{t-1}x\|_2 \ \le\ C(s,\alpha) e^{c(\alpha)} e^{-tc_\alpha}\, \|x\|_2 .$$

The assertion follows. ∎

Remark 6.6. In case of the free group and $\psi(g) = |g|$ we have $\alpha = 1$ and $s = 1$. This yields the order $t^{-3/2}$ and hence property (R_6). According to Varopoulos' definition [Var85] this means dimension $d = n/2 = 3$, as predicted by P. Biane [Bia].

Remark 6.7. According to the work of Rieffel and Ozawa[OR05] hyperbolic groups satisfies rapid decay with $s = 1$ and $d = 3$.

Proof of Corollary 0.2. According to Lemma 6.5 the assumption ii) of Theorem 6.3 are satisfied for $\mathcal{A} = \mathbb{C}[G]$. Since G is finitely generated we know that that the span F_k of work of length k are finite dimensional. By assumption the inverse of the operator $A(\lambda(g)) = \psi(g)\lambda(g)$ satisfies $\|A^{-1} : F_k \to F_k\| \le c_\alpha^{-1} k^{-\alpha}$ and hence A^{-1} is compact on $L_2^0(N)$. This provides assumption i) and Theorem 6.3 implies the assertion. ∎

Example 6.8. 1) The most natural examples are cocompact lattices $\Gamma \subset G$, where

$$G \in \{SO_0(n,1), SU(n,1)\} .$$

Let us indicate that the assumptions are verified for $\alpha = 1$. Indeed, we first recall that G acts on a hyperbolic space X and isometrically on the virtual boundary ∂X. Moreover, there exists a quadratic form Q on the boundary such that

$$\phi(d(x,y)) \ =\ Q(\mu_x - \mu_y)$$

holds for all $x, y \in \partial x$. Here d is the hyperbolic distance and $\phi(r)$ behaves like $2\log\cosh(r)$ for large r. This means $c_1 r \le \phi(r) \le c_2 r$. By the Milnor-Swarc Lemma (see e.g. [Roe03]), we also know that for cocompact discrete lattice the word length is quasi isometric to hyperbolic distance

$$c_1^{-1} l(g) \ \le\ d(gx_0, x_0) \ \le\ c_2 l(g)$$

given by a fixed base point. This yields $s = 1$. Hence we find dimension 3 in all of these cases.

2) The assumptions are satisfied for the free group in finitely many generators by the work of Haagerup [Haa79] (see also [CCJ01]).

3) Let G_1 and G_2 be two groups with rapid decay and conditionally negative functions ψ_1, ψ_2 satisfying (6.2) with $\alpha = \min(\alpha_1, \alpha_2) \leq 1$. Then $\psi(g,h) = \psi_1(g) + \psi_2(h)$ also satisfies (6.2). According to Jolissaint's work [Jol90, Lemma 2.1.2], the product also has rapid decay. Thus $T_t(\lambda((g,h)) = e^{-t\psi(g,h)}\lambda(g,h)$ defines a completely positive semigroup for which the assumptions of Theorem 6.3 are also satisfied.

4) Let (G_i, l_i, ψ_i) be groups with rapid decay and conditionally length functions ψ satisfying (6.2) with parameter k_α. According to [Jol90, Theorem 2.2.2] we know that $(*_i G_i, *l_i)$ has property RD where

$$*l_i(w_1 \cdots w_n) \;=\; \sum_i |w_i|_{l_i}$$

here $w_j \in G_{i_j}$. Bożejko proved that $\psi_t(w_1 \cdots w_n) = e^{-t\sum_j \psi_{i_j}(w_j)}$ are still positive definite and hence the free sum $\psi(w_1 \cdots w_n) = \sum_j \psi_{i_j}(w_j)$ is a conditionally negative definite function on $*_i G_i$ such that

$$\psi(w_1 \cdots w_n) \;=\; \sum_j \psi_{i_j}(w_j) \;\geq\; c(\alpha) \sum_j |w_j|^\alpha \;\geq\; c(\alpha)(\sum_j |w_j|)^\alpha$$

holds for $\alpha \leq \min\{1, \alpha_j\}$. Hence the free product is again a quantum metric space.

6.3. **Torsion free ordered groups.** In this section we show that multipliers on $\mathbb{Z}$ can be used to obtain result for torsion free ordered groups. Our main application is the well-known Hilbert transform in this context of sub-diagonal von Neumann algebras. Let us consider a discrete group G with normal divisors

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots$$

such that $\bigcap_i G_i = \{1\}$ and

$$G_i/G_{i+1} \;=\; \mathbb{Z}\,. \tag{6.3}$$

It is very easy to see that if we were to have $G_i/G_{i+1} = \mathbb{Z}^{n_i}$ that the sequence can be further refined to satisfy (6.3). Our aim is to use Riesz transforms to show the boundedness of the Hilbert transform for ordered groups. Let us recall that in the situation above the cone of positive group elements P is given by

$$P \;=\; \{g \in G : g \in G_i \setminus G_{i+1} \text{ and } gG_{i+1} \;\geq\; 0\}\,.$$

Clearly, the integer i is uniquely determined by g. Here "$\;\geq 0$" is the usual relation in $\mathbb{Z}$. We have $P \cup P^{-1} = G \setminus \{1\}$. In the following we denote by $(VN(G_i))_{i\,\geq\,0}$ the reversed martingale filtration given by the conditional expectation $E_i(\lambda(g)) = 1_{g \in G_i}\lambda(g)$.

Definition 6.9. *Let $\bigcup_k N_k \subset N$ be a weakly dense martingale filtration. A* tangent dilation *is given by a von Neumann algebra M and trace (state and modular group) preserving homomorphisms $\pi_k : N_k \to M$, $\rho : N \to M$ such that*

i) *The conditional expectation $E_\rho : M \to \rho(N)$ satisfies*

$$\rho E_{k-1} \;=\; E_\rho \pi_k$$

for all k.

ii) *The von Neumann algebras $M_k = \pi_k(N_k)$ are successively independent over $\rho(N)$.*

Lemma 6.10. *Let (N_k) be a martingale filtration and $(\rho, (\pi_k)_k)$ a tangent dilation. Let $1 < p < \infty$. Then the map $d : L_p(N) \to L_p(M)$ given by*

$$dx \;=\; \sum_k \pi_k(d_k(x))$$

gives a linear isomorphic embedding. In the limit cases d is bounded between the corresponding martingale BMO and H_1 spaces.

Proof. Let us first consider $p \geq 2$. We apply the Rosenthal inequality and deduce that

$$\begin{aligned}\|dx\|_p \;&\leq\; cp \left(\left(\sum_k \|\pi_k(d_k(x))\|_p^p \right)^{\frac{1}{p}} + \|(\sum_k E_\rho(\pi_k(|d_k(x)|^2 + |d_k(x)^*|^2))^{1/2}\|_p \right) \\ &\leq cp \left(\left(\sum_k \|d_k(x)\|_p^p \right)^{\frac{1}{p}} + \|(\sum_k E_{k-1}(|d_k(x)|^2 + |d_k(x)^*|^2))^{1/2}\|_p \right)\end{aligned}$$

Therefore the Burkholder/Rosenthal inequality implies

$$\|dx\|_p \;\leq\; cp\, c(p)\|x\|_p \,.$$

A similar argument applies for the BMO norms. Indeed, we denote by $\hat{E}_k$ the conditional expectation onto the von Neumann algebra $\hat{M}_k$ generated by $\rho(N_k)$ and $\pi_1(N_1), ..., \pi_k(N_k)$. Then we deduce from being successively independent that

$$\begin{aligned}\|dx\|_{BMO_c} &= \sup_n \|\hat{E}_n(\sum_{k\,\geq\, n} \pi_k(d_k(x)^2))\| \sim_2 \|\pi_k(d_k(x)^2)\| + \|\sum_{k>n} \hat{E}_n\hat{E}_{k-1}(\pi_k(d_k(x)^2))\| \\ &= \|\pi_k(d_k(x)^2)\| + \|\sum_{k>n} \hat{E}_n\hat{E}_\rho(\pi_k(d_k(x)^2))\| \\ &= \|\pi_k(d_k(x)^2)\| + \|\sum_{k>n} \rho(\sum_k E_{k-1}(d_k(x)^2))\| \sim_2 \|x\|_{BMO_c} \,.\end{aligned}$$

Therefore d yields an isomorphic embedding $d : BMO_{c/r}(N_k) \to BMO_{c/r}(\hat{M}_k)$. For $1 \leq p \leq 2$, we see that similarly d is bounded on h_p^d, h_p^c and h_p^r. However, for $1 \leq p < 2$, we know that $H_p^c = h_p^d + h_p^c$ holds with equivalent norms and hence d is also continuous on H_p^c. Using $tr(dx^*dy) = tr(x^*y)$ it then follows that d is an isomorphism on H_p^c and H_p^r for all $1 \leq p < \infty$ and on BMO_c, BMO_r. The assertion follows. ∎

Lemma 6.11. *For a group $G = G_0 \trianglerighteq G_1 \cdots$ there is a canonical tangent dilation.*

Proof. Let $\tilde{G} = G \times \mathbb{Z}$. Then we define

$$\pi_k : G_k \to G \times \mathbb{Z} \,,\; \pi(g) \;=\; (g, gG_{k+1}) \,.$$

Let $E : VN(\tilde{G}) \to VN(\tilde{G})$ be the conditional expectation onto $VN(G)$ and ρ the canonical embedding . Then clearly,

$$E(\pi_k(g)) \;=\; \begin{cases} g & g \in G_{k+1} \\ 0 & \text{else} \end{cases} \,.$$

This means $E(\pi_k(g)) \;=\; \rho(E_{G_{k+1}}(g))$. Note here that for a reversed filtration the definition of tangent filtration has to be suitably modified. Finally, let $\tilde{G}_k \subset \tilde{G}$ be the subgroup generated by $\rho(G_k)$ and $\mathbb{Z}$. Let $g \in G_{k-1}$. Then we have

$$E_{\tilde{G}_k}(\pi_{k-1}(g)) \;=\; 1_{g\in G_k}(g)(g, gG_k) \;=\; E(\pi_{k-1}(g)) \,,$$

because only for $g \in G_k$ we have a non-trivial term. ∎

In the following we consider the Hilbert transform

$$H(g) \;=\; i(1_P(g) - 1_P(g^{-1}))g$$

induced by the order. Note that

$$H(g)^* \;=\; H(g^{-1})\,.$$

Lemma 6.12. *Let $P_t^{\mathbb{Z}}$ be the Poisson semigroup on $VN(\mathbb{Z})$ and $P_t = id \otimes P_t^{\mathbb{Z}}$ the Poisson semigroup with generator A and gradient form Γ. Then*

$$\Gamma(dHx, dHx) \;=\; \Gamma(dx, dx)$$

and

$$P_t|dHx|^2 - |P_t dHx|^2 \;=\; P_t|dx|^2 - |P_t dx|^2$$

holds for all $x \in \mathbb{C}[G]$.

Proof. For the first assertion we consider $g, h \in G$ and i, j such that $g_1 \in G_i \backslash G_{i+1}$, $g_2 \in G_j \backslash G_{j+1}$. If $g_1 G_{i+1} \;\geq\; 0$ and $g_2 G_{j+1} \;\geq\; 0$ or $g_1 G_{i+1} \leq 0$ and $g_2 G_{j+1} \;\geq\; 0$ we have

$$\Gamma(dHg_1, dHg_2) \;=\; \Gamma(dg_1, dg_2)\,.$$

The interesting case is given by $k = g_1 G_{i+1} \;\geq\; 0$ and $j = g_2 G_{i+1} \leq 0$. Then we note that

$$\Gamma(\lambda(k), \lambda(j)) \;=\; \frac{|k| + |j| - |k-j|}{2} \lambda(k)^* \lambda(j) \;=\; 0\,.$$

The second assertion follows similarly in this case

$$P_t^{\mathbb{Z}}(\lambda(k)^*\lambda(j)) - P_t^{\mathbb{Z}}(\lambda(k))^* P_t^{\mathbb{Z}}(\lambda(j)) \;=\; (e^{-t|j-k|} - e^{-t|k|} e^{-t|j|})\lambda(k)^*\lambda(j) \;=\; 0\,.$$

Thus the sign change only occurs when we have a 0-coefficient. ■

Theorem 6.13. *H is bounded on $L_p(VN(G))$ for all $1 < p < \infty$.*

Proof. Let x be selfadjoint. Theorem 2.9iii) and H^∞-calculus implies

$$\|dx\|_p \sim \|(\int \Gamma(P_s dx, P_s dx) ds)^{1/2}\| \;=\; \|(\int \Gamma(P_s dHx, P_s dHx) ds)^{1/2}\| \sim \|Hx\|_p\,.$$

Here we use that for selfadjoint x the element Hx is also selfadjoint. For an alternative proof, we can use interpolation and note that $dH : VN(G) \to BMO(\tilde{G})$ is bounded and $dH : L_2(VN(G)) \to L_2(VN(\tilde{G}))$ and hence

$$dH : L_p(VN(G)) \to L_p(VN(\tilde{G}))$$

is bounded. Finally, the assertion follows from Lemma 6.10

$$\|dHx\|_p \sim_{c(p)} \|Hx\|_p$$

This yields the assertion for $2 \leq p < \infty$. Duality implies the result for $1 < p \leq 2$ as well. ■

Remark 6.14. With the help of the tangent dilation, we can show that every completely bounded Fourier multiplier on $\mathbb{Z}$ induces a Fourier multiplier on $VN(G)$, by applying it to $d(x)$. We can even use different multipliers on $\mathbb{Z}^\infty$ by modifying $d(g) \;=\; (g, ..., gG_{k+1}, ...)$ for $g \in G_k \setminus G_{k+1}$.

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